$\lim_{x \to \infty} \int_2^3 \frac{1}{dx} dy$

THE DEFINITION OF THE LIMIT

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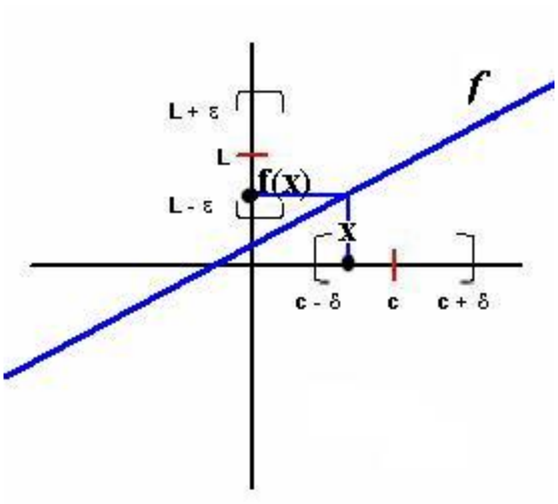
So far we have found limits by looking at graphs of "mild-mannered" functions. However, most graphs cannot be produced reliably enough to say with 100% certainty that a limit exists or does not exist.

To ensure that we have actually found the correct limit from a graph, a rigorous definition was developed that would allow us to determine if our limit guess is correct or incorrect.

Using the picture below as a reference, we will now state the **Definition of the Limit**. It is used to prove whether or not a particular limit statement is true or false.



Formally, the **Definition of the Limit** is written as follows:

$\lim_{x \to c} f(x) = L$, if the following criterion holds:

Given any radius $\varepsilon > 0$ about the number ***L***, there exists a radius $\delta > 0$ about the number ***c*** such that

$0 < |x - c| < \delta$ implies $|f(x) - L| < \varepsilon$

The distance between any number ***x*** and ***c*** is less than δ, but greater than ***0***.	The distance between the corresponding number ***f (x)*** and ***L*** is less than ε.

This means, for all x-values within any radius $\delta > 0$ about ***c***, there must be corresponding y-values within a radius $\varepsilon > 0$ about ***L***.

NOTE: δ is pronounced *delta* and ε is pronounced *epsilon*!

Strategy for proving that a limit statement is true or false.

Step 1:

Solve the inequality $|f(x)-L|<\varepsilon$ to find an interval about the number ***c***.

Step 2:

Find the radius $\delta>0$ by calculating the shortest distance from ***c*** to the endpoints of the interval found in Step 1.

Problem 1:

Prove that the statement $\lim_{x\to 1}(-4x)=-4$ is true by using $\varepsilon=1$ and finding an appropriate $\delta>0$.

Step 1 (find $|f(x)-L|<\varepsilon$!!!):

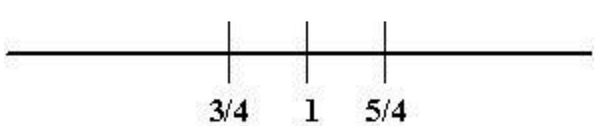
$$|-4x-(-4)|<1$$
$$-1<-4x+4<1$$
$$-5<-4x<-3$$
$$\frac{5}{4}>x>\frac{3}{4}$$

or $\frac{3}{4}<x<\frac{5}{4}$

We have found the interval $\left(\frac{3}{4},\frac{5}{4}\right)$ about $c=1$.

Step 2:

Now we need to find the radius $\delta>0$ by calculating the shortest distance from $c=1$ to the endpoints of the interval $\left(\frac{3}{4},\frac{5}{4}\right)$.



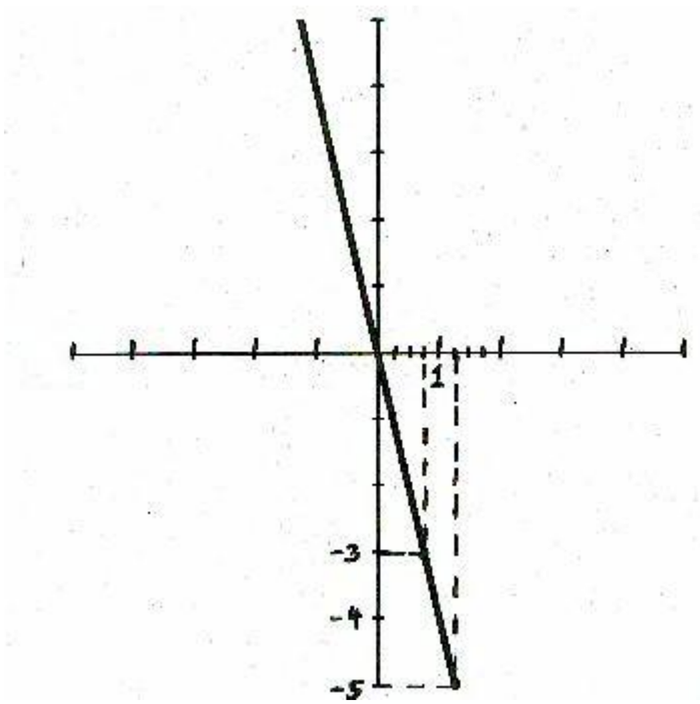
$\frac{5}{4} - 1 = \frac{1}{4}$ and $1 - \frac{3}{4} = \frac{1}{4}$

We find that $\delta = \frac{1}{4}$.

Since we were able to find a δ for a given ε, we proved that as we approach ***1*** on the x-axis, we do approach ***- 4*** on the y-axis.

NOTE: We showed in effect, that all x-values in the interval ***(3/4, 5/4)*** indeed have their y-values in the interval ***(-5, -3)***.

Please investigate the graph below!



Problem 2:

Prove that the statement $\lim_{x \to 6} (9 - \frac{1}{6}x) = 8$ is true by using any $\varepsilon > 0$ and finding the appropriate $\delta > 0$.

Step 1:

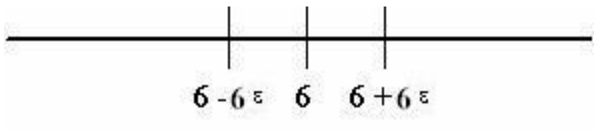
$$|(9 - \tfrac{1}{6}x) - 8| < \varepsilon$$
$$-\varepsilon < 1 - \tfrac{1}{6}x < \varepsilon$$
$$-1 - \varepsilon < -\tfrac{1}{6}x < -1 + \varepsilon$$
$$6 + 6\varepsilon > x > 6 - 6\varepsilon$$

or $6 - 6\varepsilon < x < 6 + 6\varepsilon$

We have found the interval $(6-6\varepsilon, 6+6\varepsilon)$ about $c=6$.

Step 2:

Now we need to find the radius $\delta>0$ by calculating the shortest distance from $c=6$ to the endpoints of the interval $(6-6\varepsilon, 6+6\varepsilon)$.



$6+6\varepsilon-6=6\varepsilon$ AND $6-(6-6\varepsilon)=6\varepsilon$

We find that $\delta=6\varepsilon$.

Since we were able to find a δ for a given ε, we proved that as we approach 6 on the x-axis, we do approach 8 on the y-axis.

Problem 3:

Prove that the statement $\lim_{x\to 5}\sqrt{x-1}=2$ is true by using $\varepsilon=1$ and finding the appropriate $\delta>0$.

Step 1:

$$|\sqrt{x-1}-2|<1$$
$$-1<\sqrt{x-1}-2<1$$
$$1<\sqrt{x-1}<3$$

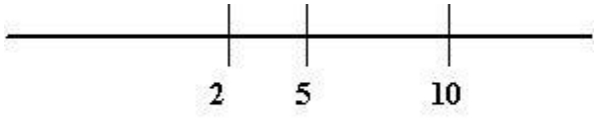
now we'll square all areas to get

$$1<x-1<9$$
$$2<x<10$$

We have found the interval $(2,10)$ about $c=5$.

Step 2:

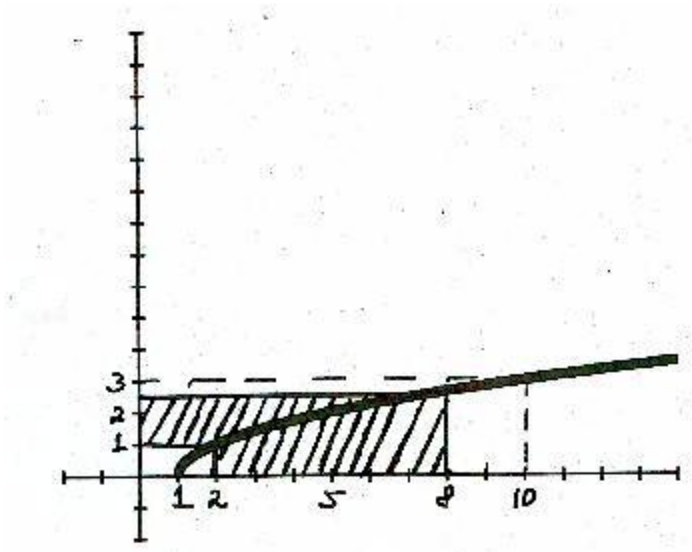
Now we need to find the radius $\delta>0$ by calculating the shortest distance from $c=5$ to the endpoints of the interval $(2,10)$.



$10-5=5$ and $5-2=3$

We find that $\delta=3$.

Since we were able to find a δ for a given ε, we proved that as we approach ***5*** on the x-axis, we do approach ***2*** on the y-axis.



Problem 4:

Prove that the statement $\lim_{x\to 2}(x^2)=4$ is true by using $\varepsilon=1$ and finding the appropriate $\delta>0$.

Step 1:

$$|x^2-4|<1$$
$$-1<x^2-4<1$$
$$3<x^2<5$$

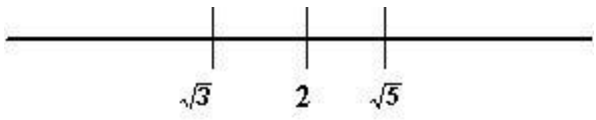
now we'll use the Square Root Property to get

$\sqrt{3}<x<\sqrt{5}$ or $-\sqrt{5}<x<-\sqrt{3}$

Since we are approaching a positive number we will use $\sqrt{3}<x<\sqrt{5}$ and we have found the interval $(\sqrt{3},\sqrt{5})$ about $c=2$.

Step 2:

Now we need to find the radius $\delta > 0$ by calculating the shortest distance from $c = 2$ to the endpoints of the interval $\left(\sqrt{3}, \sqrt{5}\right)$.



$\sqrt{5} - 2 \approx 0.2361$ and $2 - \sqrt{3} \approx 0.2679$

We find that $\delta = \sqrt{5} - 2$.

Since we were able to find a δ for a given ε, we proved that as we approach ***2*** on the x-axis, we do approach ***4*** on the y-axis.

Problem 5:

Prove that the statement $\lim_{x \to 1/2} \left(\frac{1}{x}\right) = 2$ is true by using $\varepsilon = 0.01$ and finding the appropriate $\delta > 0$.

Step 1:

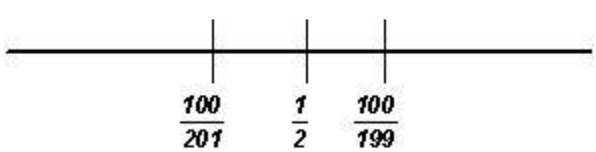
$$\left|\tfrac{1}{x} - 2\right| < 0.01$$
$$-0.01 < \tfrac{1}{x} - 2 < 0.01$$
$$1.99 < \tfrac{1}{x} < 2.01$$

Now we'll invert all areas to get

$$\frac{1}{2.01} < x < \frac{1}{1.99}$$

Step 2:

Now we need to find the radius $\delta > 0$ by calculating the shortest distance from $c = \frac{1}{2}$ to the endpoints of the interval $\left(\frac{100}{201}, \frac{100}{199}\right)$.



$$\frac{100}{199} - \frac{1}{2} = \frac{1}{398} \approx 0.002513$$ and

$$\frac{1}{2} - \frac{100}{201} = \frac{1}{402} \approx 0.002488$$

We find that $\delta = \frac{1}{402}$.

Since we were able to find a δ for a given ε, we proved that as we approach ***1/2*** on the x-axis, we do approach ***2*** on the y-axis.

Problem 6:

Prove that the statement $\lim_{x \to 3} (4x - 5) = 10$ is false by using $\varepsilon = 1$.

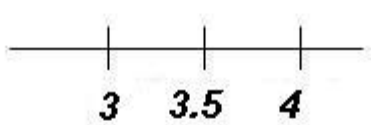
Step 1:

$$|(4x - 5) - 10| < 1$$
$$-1 < 4x - 15 < 1$$
$$14 < 4x < 16$$
$$3.5 < x < 4$$

It looks like we have found the interval $(3.5, 4)$ about $c = 3$.

Step 2:

Now we must see if we can find a radius $\delta > 0$ by calculating the shortest distance from $c = 3$ to the endpoints of the interval $(3.5, 4)$.



We see immediately that we did not find a radius $\delta > 0$ about $c = 3$. Therefore, the statement $\lim_{x \to 3} (4x - 5) = 10$ must be false.

Problem 7:

Show that the statement $\lim_{x \to c} x = c$ is true for any $\varepsilon > 0$.

Step 1:

$$|x - c| < \varepsilon$$
$$-\varepsilon < x - c < \varepsilon$$
$$c - \varepsilon < x < c + \varepsilon$$

We have found the interval $(c - \varepsilon, c + \varepsilon)$ about c.

Step 2:

Now we need to find a radius $\delta > 0$ by calculating the shortest distance from c to the endpoint of the interval $(c - \varepsilon, c + \varepsilon)$.

That is, $c + \varepsilon - c = \varepsilon$ and $c - (c - \varepsilon) = \varepsilon$

We find that $\delta = \varepsilon$.

Since we were able to find a $\delta > 0$ for any $\varepsilon > 0$, we can confirm that the statement $\lim_{x \to c} x = c$ is true.

Problem 8:

Show that the statement $\lim_{x \to c} b = b$, where b is any real number, is true for any $\varepsilon > 0$.

Step 1:

$$|b - b| < \varepsilon$$
$$0 < \varepsilon$$

But this inequality is always true, no matter what value we choose for $\delta > 0$.

Therefore, we can confirm that the statement $\lim_{x \to c} b = b$ is true.