CRITICAL NUMBERS

Prepared by Ingrid Stewart, Ph.D., College of Southern Nevada
Please Send Questions and Comments to ingrid.stewart@csn.edu. Thank you!

Definition of Critical Numbers

Critical numbers are values of x at which $f'(x) = 0$ or $f'(x)$ does not exist.

How to find *Critical Numbers*

1. Set the **first** derivative equal to 0 and solve.
2. If the **first** derivative has a denominator containing a variable, set the denominator equal to 0 and solve.

NOTE:

Solutions derived from Steps 1 and 2 must be in the domain of the function to be considered *critical numbers*.

Problem 1:

Find the *critical numbers* of $f(x) = 2x^3 + x^2 - 20x + 4$ with domain $(-\infty, \infty)$

The first derivative is $f'(x) = 6x^2 + 2x - 20$.

Step 1 - Set the first derivative equal to 0

$$0 = 6x^2 + 2x - 20$$
$$(2x + 4)(3x - 5) = 0$$

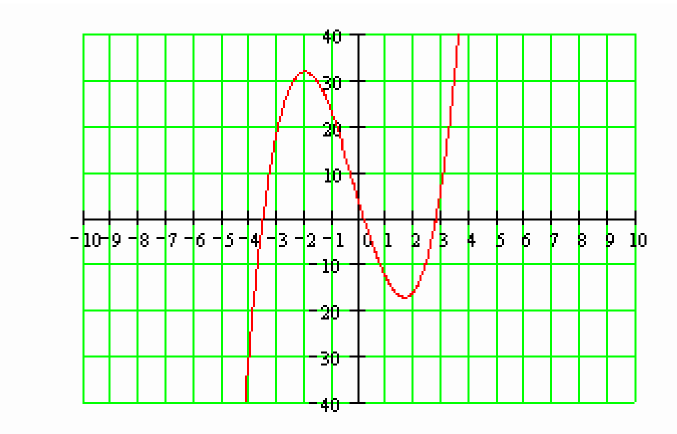
and by the *Zero Product Principle* $x = -2$ or $x = \frac{5}{3}$.

Both -2 and $\frac{5}{3}$ are *critical numbers* because they are in the domain of the function.

Step 2 - Not possible, since the first derivative does not have a denominator containing a variable.

Therefore, the function has two *critical numbers*, that is, -2 and $\frac{5}{3}$.

Below is the graph of the function. The *critical numbers* are values found along the x-axis. Please note that for this function the *critical numbers* seem to be the x-coordinates of the peak and valley of the graph of the function.



Problem 2:

Find the *critical numbers* of $f(x) = x^{2/3} - 1$ with domain $(-\infty, \infty)$

The first derivative is $f'(x) = \frac{2}{3}x^{-1/3} = \dfrac{2}{3x^{1/3}}$.

Please note that it is BEST to write the derivative as one single fraction with all positive exponents!

Step 1 - Set the first derivative equal to 0

$0 = \dfrac{2}{3x^{1/3}}$ and multiplying both sides by the denominator indicates that there is no solution. Note $0 \neq 2$! Therefore, this step did not produce any *critical numbers*.

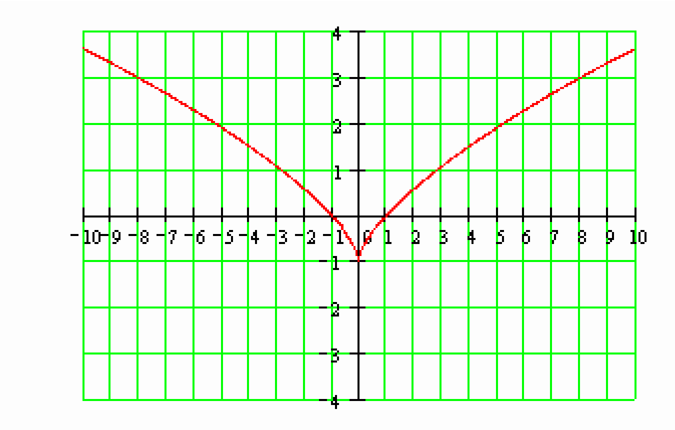
Step 2 - Set the denominator of the first derivative equal to 0

$3x^{1/3} = 0$ and $x = 0$

0 is a *critical number* because it is in the domain of the function.

Therefore, the function has one *critical number*, that is, 0.

Below is the graph of the function. The *critical numbers* are values found along the x-axis. Please note that for this function the *critical number again* seems to be the x-coordinate of the valley of the graph of the function. However, the bottom of this valley is not rounded but consists of a sharp point, which is called a cusp.



Problem 3:

Find the *critical numbers* of $f(x) = \sqrt[3]{x^2 - x - 2}$ with domain $(-\infty, \infty)$

The first derivative is

$$f'(x) = (2x-1)[\tfrac{1}{3}(x^2 - x - 2)^{-2/3}] = \frac{2x-1}{3(x^2-x-2)^{2/3}}.$$

Step 1 - Set the first derivative equal to ***0***

$$0 = \frac{2x-1}{3(x^2-x-2)^{2/3}}$$

and multiplying both sides by the denominator indicates that

$$0 = 2x - 1$$
$$x = \tfrac{1}{2}$$

$\frac{1}{2}$ is a *critical number* because it is in the domain of the function.

Step 2 - Set the denominator of the first derivative equal to ***0***

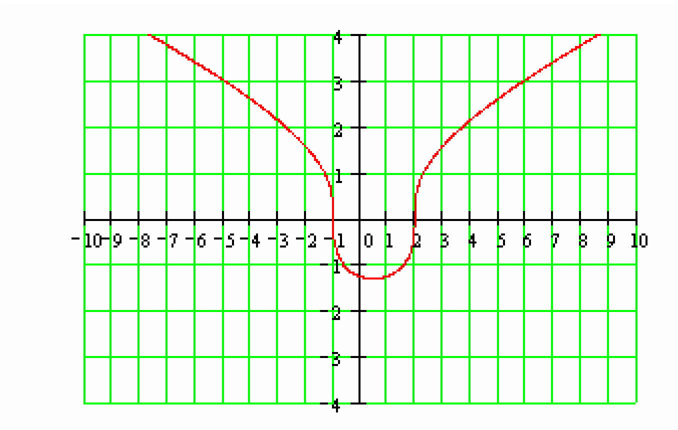
$$3(x^2 - x - 2)^{2/3} = 0$$
$$(x^2 - x - 2)^{2/3} = 0$$
$$[(x^2 - x - 2)^{2/3}]^{3/2} = (0)^{3/2}$$
$$x^2 - x - 2 = 0$$
$$(x-2)(x+1) = 0$$

and by the *Zero Product Principle* $x = 2$ or $x = -1$.

Both 2 and -1 are *critical numbers* because they are in the domain of the function.

Therefore, the function has three *critical numbers*, that is, -1, $\frac{1}{2}$, and 2.

Below is the graph of the function. The *critical numbers* are values found along the x-axis. Please note that for this function one of the *critical numbers* seems to be the x-coordinate of the valley of the graph of the function. The other two seem to be x-intercepts.



Problem 4:

Find the *critical numbers* of $f(x) = x\sqrt{9 - x^2}$ with domain $[-3,3]$

The first derivative is

$$f'(x) = (9 - x^2)^{\frac{1}{2}} + x(-2x)[\tfrac{1}{2}(9 - x^2)^{-\frac{1}{2}}]$$
$$= (9 - x^2)^{\frac{1}{2}} - \frac{x^2}{(9 - x^2)^{\frac{1}{2}}}$$
$$= \frac{9 - x^2 - x^2}{(9 - x^2)^{\frac{1}{2}}}$$.

and
$$f'(x) = \frac{9 - 2x^2}{(9 - x^2)^{\frac{1}{2}}}$$

Step 1 - Set the first derivative equal to 0

$$0 = \frac{9 - 2x^2}{(9 - x^2)^{\frac{1}{2}}}$$

and multiplying both sides by the denominator indicates that

$$0 = 9 - 2x^2$$
$$x^2 = \frac{9}{2}$$

and by the *Square Root Property* we find that $x = -\frac{3}{\sqrt{2}} = -\frac{3\sqrt{2}}{2}$ and $x = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$
.

Both $-\frac{3\sqrt{2}}{2} \approx -2.12$ and $\frac{3\sqrt{2}}{2} \approx 2.12$ are *critical numbers* because they are in the domain of the function.

Step 2 - Set the denominator of the first derivative equal to 0

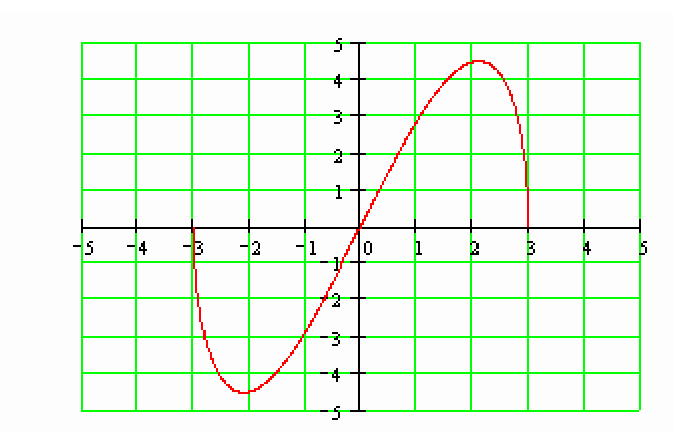
$$(9 - x^2)^{1/2} = 0$$
$$[(9 - x^2)^{1/2}]^2 = (0)^2$$
$$9 - x^2 = 0$$
$$x^2 = 9$$

and by the *Square Root Property* $x = -3$ or $x = 3$.

Both -3 and 3 are *critical numbers* because they are in the domain of the function.

Therefore, the function has four *critical numbers*, that is, -3, $-\frac{3\sqrt{2}}{2} \approx -2.12$, $\frac{3\sqrt{2}}{2} \approx 2.12$, and 3.

Below is the graph of the function. The *critical numbers* are values found along the x-axis. Please note that for this function two of the *critical numbers* seem to be the x-coordinates of the peak and valley of the graph of the function. The other two seem to be x-intercepts.



Problem 5:

Find the *critical numbers* of $f(x) = x\sqrt{2} - 2\cos x$ with restricted domain $[-2\pi, 2\pi]$

The first derivative is $f'(x) = \sqrt{2} + 2\sin x$.

Step 1 - Set the first derivative equal to 0

$$0 = \sqrt{2} + 2\sin x$$
$$\sin x = -\tfrac{\sqrt{2}}{2}$$
$$x = \arcsin\left(-\tfrac{\sqrt{2}}{2}\right)$$

$$x = -45° \equiv -\frac{\pi}{4}$$

Reference angle is $45° \equiv \dfrac{\pi}{4}$

The numeric values of sine are negative in QIII and QIV.

Therefore, the solutions in the interval $[0, 2\pi)$ are

$225° \equiv \dfrac{5\pi}{4}$ and $315° \equiv \dfrac{7\pi}{4}$

and ALL solutions are

$\dfrac{5\pi}{4} + 2\pi k$ and $\dfrac{7\pi}{4} + 2\pi k$, where k is any integer.

NOTE: You can think of k as the number of round trips from and to the terminal side of the angles found in the interval $[0, 2\pi)$. A negative k indicates a clockwise movement (negative angles) and a positive k indicates a counter-clockwise movement (positive angles).

For example,

when $k = -1$ we get the solutions on the interval $[-2\pi, 0)$

when $k = 0$ we get the solutions on the interval $[0, 2\pi)$

when $k = 1$ we get the solutions on the interval $[2\pi, 4\pi)$

etc.

The solutions in the interval $[-2\pi, 2\pi]$ are

$\frac{5\pi}{4} + 2\pi(0) = \frac{5\pi}{4}$ and $\frac{7\pi}{4} + 2\pi(0) = \frac{7\pi}{4}$ and

$\frac{5\pi}{4} + 2\pi(-1) = \frac{-3\pi}{4}$ and $\frac{7\pi}{4} + 2\pi(-1) = \frac{-\pi}{4}$

$\frac{-3\pi}{4}$, $\frac{-\pi}{4}$, $\frac{5\pi}{4}$, and $\frac{7\pi}{4}$ are *critical numbers* because they are in the domain of the function.

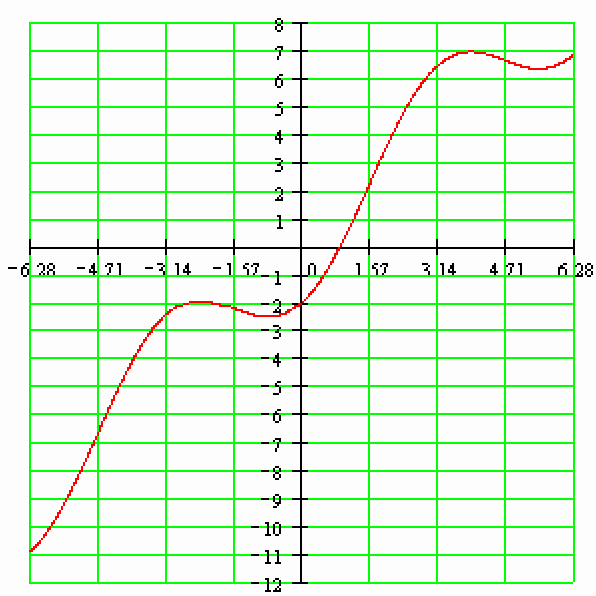
Step 2 - Not possible, since the first derivative does not have a denominator containing a variable.

Therefore, the function has four *critical numbers*, that is,

$\frac{-3\pi}{4}$**,** $\frac{-\pi}{4}$**,** $\frac{5\pi}{4}$**, and** $\frac{7\pi}{4}$**.**

Below is the graph of the function. The *critical numbers* are values found along the x-axis. Please note that for this function the *critical numbers* seem to be the x-coordinates of the peaks and valleys of the graph of the function.

$\frac{-3\pi}{4} \approx -2.356$, $\frac{-\pi}{4} \approx -0.785$, $\frac{5\pi}{4} \approx 3.927$, $\frac{7\pi}{4} \approx 5.498$



Problem 6:

Find the *critical numbers* of $f(x) = (x+2)^3 - 4$ with domain $(-\infty, \infty)$

The first derivative is $f'(x) = 3(x+2)^2$.

Step 1 - Set the first derivative equal to 0

$$0 = 3(x+2)^2$$
$$x + 2 = 0$$

and $x = -2$

-2 is a *critical number* because it is in the domain of the function.

Step 2 - Not possible, since the first derivative does not have a denominator containing a variable.

Therefore, the function has one *critical number*, that is, -2.

Below is the graph of the function. The *critical numbers* are values found along the x-axis. Please note that while this function has a *critical number,* it does not appear that it is the x-coordinate of a peak or valley. As a matter of fact, it seems to be the x-coordinate of the point at which concavity changes (see Precalculus, Chapter 2.3 and 2.4).