$$\lim_{x \to \infty} \int_2^3 \frac{1}{dx} dy$$

THE CHAIN RULE

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Composite Function Review

- Let $h(x) = \sin x$ and $g(x) = 2x$, then the *Composite Function* is $h[g(x)] = h(2x) = \sin(2x)$
- Let $h(x) = \ln x$ and $g(x) = 5x + 3$, then the Composite Function is $h[g(x)] = h(5x+3) = \ln(5x+3)$
- Let $h(x) = e^x$ and $g(x) = 2x^2 - 5$, then the Composite Function is $h[g(x)] = h(2x^2 - 5) = e^{2x^2-5}$

The Chain Rule for Differentiation

You can find the proof of this rule in the online textbook as a separate document.

If $f(x) = h[g(x)]$ is a composite of the functions ***h(x)*** and ***g(x)***, then $f'(x) = g'(x) \cdot h'[g(x)]$.

Given functions of the following form, where $u = g(x)$ is a function of x, then by the chain rule they have derivatives as shown below.

1. If $f(x) = \sin u$, then $f'(x) = u' \cos u$.
2. If $f(x) = \tan u$, then $f'(x) = u' \sec^2 u$.
3. If $f(x) = \sec u$, then $f'(x) = u' \sec u \tan u$.
4. If $f(x) = \cos u$, then $f'(x) = -u' \sin u$.
5. If $f(x) = \cot u$, then $f'(x) = -u' \csc^2 u$.
6. If $f(x) = \csc u$, then $f'(x) = -u' \csc u \cot u$.
7. If $f(x) = b^u$, then $f'(x) = u' b^u \ln b$

Special Case (most frequently used) - Natural Exponential Function:

If $f(x) = e^u$, then $f'(x) = u'e^u$

8. If $f(x) = \log_b u$, then $f'(x) = \frac{u'}{u \ln b}$

Special Case (most frequently used) - Natural Logarithm (el en):

If $f(x) = \ln u$, then $f'(x) = \frac{u'}{u}$

Please note! The trigonometric, logarithmic, and exponential differentiation rules from Unit 8 are special cases of the *Chain Rule,* where $u = x$.

The General Power Rule for Differentiation - A Special Case of the Chain Rule

Let $u = g(x)$ be a function of x. If $f(x) = u^n$, where n is any real number, then $f'(x) = u' \cdot nu^{n-1}$. The general power rule is a special case of the chain rule!

Please note! The *Simple Power Rule* is a special case of the *General Power Rule* and used as a shortcut where $u = x$.

Problem 1:

Find the derivative of $f(x) = (2x-3)^2$.

In this case, we could either use the *General Power Rule* as the primary rule or the *Simple Power Rule* discussed in a previous unit.

Method 1 - Use the *Simple Power Rule* as the primary rule

$f(x) = (2x-3)^2 = 4x^2 - 12x + 9$, then $f'(x) = 8x - 12$

Method 2 - Use the *General Power Rule* as the primary rule

We know that $f'(x) = u' \cdot nu^{n-1}$. Let's find all of the parts we need for this derivative.

That is, $u = 2x - 3$, and $u' = 2$.

$$f'(x) = 2[2(2x-3)^{2-1}] = 4(2x-3) = 8x - 12$$

NOTE: Whenever possible you should ALWAYS combine like terms in your solutions!

Problem 2:

Find the derivative of $f(x) = (2x - 3)^{50}$.

In this case, we can ONLY use the *General Power Rule*! Why?

For $u = 2x - 3$, and $u' = 2$, we find that

$$f'(x) = 2[50(2x - 3)^{49}]$$

and $f'(x) = 100(2x - 3)^{49}$

Problem 3:

Find the derivative of $f(x) = (3x - 2x^2)^5$.

For $u = 3x - 2x^2$, and $u' = 3 - 4x$, we find that $f'(x) = (3 - 4x)[5(3x - 2x^2)^4]$

and $f'(x) = 5(3 - 4x)(3x - 2x^2)^4$

NOTE: Whenever possible you should ALWAYS write constants first in your solutions!

In this case, combining like terms is more difficult than what its worth. Any power greater than 2 should be left alone!

Problem 4:

Find the derivative of $f(x) = \sqrt[3]{(x^2 - 1)^2}$. Express your answer without negative exponents!

$$f(x) = \sqrt[3]{(x^2 - 1)^2} = (x^2 - 1)^{2/3}$$

For $u = x^2 - 1$, and $u' = 2x$, we find that

$$f'(x) = 2x[\tfrac{2}{3}(x^2 - 1)^{-1/3}] = \tfrac{4}{3}x(x^2 - 1)^{-1/3}$$

and $f'(x) = \dfrac{4x}{3(x^2 - 1)^{1/3}}$

Problem 5:

Find the derivative of $g(t) = \frac{-7}{(2t-3)^2}$.

$$g(t) = \frac{-7}{(2t-3)^2} = -7(2t-3)^{-2}$$

In order to differentiate any fraction, we MUST use the *Quotient Rule* or the *Product Rule* or sometimes the *Constant Multiple Rule*. While any fraction can be changed into a product, it is often only advisable to do so when the numerator is a constant so that the *Constant Multiple Rule* can be used.

Since the numerator in the function above is a constant, we MUST either use the *Quotient Rule* as the primary rule or the *Constant Multiple Rule*.

Method 1 - Use the *Quotient Rule* as the primary rule

Let $u = -7$ and $v = (2t-3)^2$, then $u' = 0$ and $v' = 2[2(2t-3)] = 4(2t-3)$. **Please note that we had to use the *General Power Rule* to find v'!**

$$g'(t) = \frac{0(2t-3)^2 - (-7)[4(2t-3)]}{[(2t-3)^2]^2}$$

$$g'(t) = \frac{28(2t-3)}{(2t-3)^4}$$

and $g'(t) = \frac{28}{(2t-3)^3}$

NOTE: Whenever possible you should ALWAYS reduce your solutions to lowest terms! However, it is not necessary to multiply out the denominator!

Method 2 - Use the *Constant Multiple Rule* as the primary rule

Using $g(t) = -7(2t-3)^{-2}$, the primary rule is the *Constant Multiple Rule* followed by the *General Power Rule*.

To differentiate $(2t-3)^{-2}$, we'll use $u = 2t-3$, and $u' = 2$,

to find $2[-2(2t-3)^{-3}]$.

Then using the *Constant Multiple Rule* we get

$$g'(t) = -7\{2[-2(2t-3)^{-3}]\} = 28(2t-3)^{-3}$$

and $$g'(t) = \frac{28}{(2t-3)^3}$$

It is not necessary to multiply out the denominator!

Problem 6:

Find the slope of the line tangent to any point on the graph of $f(x) = x^2\sqrt{1-x^2}$.

$$f(x) = x^2\sqrt{1-x^2} = x^2(1-x^2)^{1/2}$$

We must use the *Product Rule* as the primary rule.

Let $u = x^2$ and $v = (1-x^2)^{1/2}$, then $u' = 2x$ and $v' = -2x[\frac{1}{2}(1-x^2)^{-1/2}] = -x(1-x^2)^{-1/2}$. **Please note that we had to use the *General Power Rule* to find v'!**

$$\begin{aligned} f'(x) &= 2x(1-x^2)^{1/2} + x^2[-x(1-x^2)^{-1/2}] \\ &= 2x(1-x^2)^{1/2} - x^3(1-x^2)^{-1/2} \end{aligned}$$

Writing the derivative without negative exponents, we get

$$f'(x) = 2x(1-x^2)^{1/2} - \frac{x^3}{(1-x^2)^{1/2}}$$

and finally writing the derivative as a single fraction, we get

$$f'(x) = \frac{2x(1-x^2)^{1/2}(1-x^2)^{1/2} - x^3}{(1-x^2)^{1/2}} = \frac{2x(1-x^2) - x^3}{(1-x^2)^{1/2}}$$

and $$f'(x) = \frac{2x - 3x^3}{(1-x^2)^{1/2}}$$

Please note that $(1-x^2)^{1/2} \neq 1^{1/2} + (x^2)^{1/2}$.

Problem 7:

Find the derivative of $g(x) = \dfrac{x}{\sqrt[3]{x^2+4}}$ using the *Quotient Rule*. Write your answer without negative exponents.

$$g(x) = \frac{x}{\sqrt[3]{x^2+4}} = \frac{x}{(x^2+4)^{1/3}}$$

Let $u = x$ and $v = (x^2+4)^{1/3}$, then $u' = 1$ and $v' = 2x[\frac{1}{3}(x^2+4)^{-2/3}] = \frac{2}{3}x(x^2+4)^{-2/3}$. **Please note that we had to use the *General Power Rule* to find v'!**

$$g'(x) = \frac{(x^2+4)^{1/3} - x[\frac{2}{3}x(x^2+4)^{-2/3}]}{[(x^2+4)^{1/3}]^2} = \frac{(x^2+4)^{1/3} - \frac{2}{3}x^2(x^2+4)^{-2/3}}{(x^2+4)^{2/3}}$$

To get rid of the negative exponents, we'll multiply by the number ***1*** in the form $\dfrac{3(x^2+4)^{2/3}}{3(x^2+4)^{2/3}}$.

That is, $g'(x) = \dfrac{(x^2+4)^{1/3} - \frac{2}{3}x^2(x^2+4)^{-2/3}}{(x^2+4)^{2/3}} \cdot \dfrac{3(x^2+4)^{2/3}}{3(x^2+4)^{2/3}}$,

then $g'(x) = \dfrac{(x^2+4)^{1/3} \cdot 3(x^2+4)^{2/3} - \frac{2}{3}x^2(x^2+4)^{-2/3} \cdot 3(x^2+4)^{2/3}}{(x^2+4)^{2/3} \cdot 3(x^2+4)^{2/3}}$

Simplifying, we get $g'(x) = \dfrac{3(x^2+4) - 2x^2}{3(x^2+4)^{4/3}} = \dfrac{3x^2+12-2x^2}{3(x^2+4)^{4/3}}$

and $g'(x) = \dfrac{x^2+12}{3(x^2+4)^{4/3}}$

Problem 8:

Find the derivative of $g(t) = \left(\frac{3t-1}{t+3}\right)^2$.

$$g(t) = \left(\frac{3t-1}{t+3}\right)^2 = \frac{(3t-1)^2}{(t+3)^2}$$

You can differentiate this function in two ways. That is, you can use the *General Power Rule* as the primary rule or you can use the *Quotient Rule* as the primary rule.

Please note that within the *General Power Rule* the *Quotient Rule* must be used and with the *General Power Rule* the *Quotient Rule* must be used.

Method 1 - Use the *General Power Rule* as the primary rule!

In order to use the *Constant Multiple Rule* we must change the function as follows:

Then, for $u = \frac{3t-1}{t+3}$, and $u' = \frac{10}{(t+3)^2}$.

Please note that we had to use the *Quotient Rule* to find u'!

That is, given $y = \frac{3t-1}{t+3}$, we use $u = 3t-1$ and $v = t+3$, then $u' = 3$ and $v' = 1$.

Then $\frac{dy}{dx} = \frac{3(t+3)-(3t-1)(1)}{(t+3)^2} = \frac{10}{(t+3)^2} = u'$

$$g'(t) = \frac{10}{(t+3)^2}\left[2\left(\frac{3t-1}{t+3}\right)\right]$$

and $g'(t) = \frac{20(3t-1)}{(t+3)^3}$

NOTE: Whenever possible you should ALWAYS combine like terms in your solutions! However, it is not necessary to multiply out the denominator!

Method 2 - Use the *Quotient Rule* as the primary rule

Given $g(t) = \frac{(3t-1)^2}{(t+3)^2}$, let $u = (3t-1)^2$ and $v = (t+3)^2$, then $u' = 3[2(3t-1)] = 6(3t-1)$ and $v' = 2(t+3)$. **Please note that we had to use the *General Power Rule* to find both u' and v'!**

$$g'(t) = \frac{6(3t-1)(t+3)^2 - (3t-1)^2[2(t+3)]}{[(t+3)^2]^2}$$
$$= \frac{6(3t-1)(t+3)^2 - 2(3t-1)^2(t+3)}{(t+3)^4}$$

NOTE: Whenever possible you should ALWAYS combine like terms in your solutions and reduce to lowest terms! However, it is not necessary to multiply out the denominator!

However, in this case it is easier to notice that the two terms in the numerator have the factors $(3t-1)(t+3)$ in common. We will now factor them out as follows:

$$g'(t) = \frac{(3t-1)(t+3)[6(t+3) - 2(3t-1)]}{(t+3)^4}$$
$$= \frac{(3t-1)(6t+18-6t+2)}{(t+3)^3}$$

Notice that it was easier to combine like terms in the brackets than trying to combine like terms before factoring out the common factors.

Finally, we get $g'(t) = \dfrac{20(3t-1)}{(t+3)^3}$

Problem 9:

Differentiate $y = \sec^5 x$.

NOTE: $y = \sec^5 x = (\sec x)^5$

Using the *General Power Rule* with $u = \sec x$ and $u' = \sec x \tan x$, we find

$$\frac{dy}{dx} = \sec x \tan x[5(\sec x)^4]$$

and $\dfrac{dy}{dx} = 5\sec^5 x \tan x$

NOTE: Whenever possible you should ALWAYS write constants first in your solutions!

Problem 10:

Differentiate $y = \tan^2 x$.

NOTE: $y = \tan^2 x = (\tan x)^2$

Using the *General Power Rule* with $u = \tan x$ and $u' = \sec^2 x$, we find $\frac{dy}{dx} = \sec^2 x[2(\tan x)]$

and $\frac{dy}{dx} = 2\sec^2 x \tan x$

NOTE: Whenever possible you should ALWAYS write constants first in your solutions!

Problem 11:

Differentiate $y = \tan 3x$.

Using the *Chain Rule* with $u = 3x$ and $u' = 3$, we find $\frac{dy}{dx} = 3\sec^2 3x$

Problem 12:

Differentiate $y = \cot \pi x$.

Using the *Chain Rule* with $u = \pi x$ and $u' = \pi$, we find $\frac{dy}{dx} = -\pi \csc^2 \pi x$

Problem 13:

Differentiate $y = \sin(2x+3)^4$.

Using the *Chain Rule* with $u = (2x+3)^4$ and $u' = 2[4(2x+3)^3] = 8(2x+3)^3$,

we find $\frac{dy}{dx} = 8(2x+3)^3 \cos(2x+3)^4$

Problem 14:

Differentiate $y = \csc 5x^2$.

Using the *Chain Rule* with $u = 5x^2$ and $u' = 10x$, we find $\frac{dy}{dx} = -10x \csc 5x^2 \cot 5x^2$

Problem 15:

Differentiate $y = (\csc 5x)^2$.

In this case, the primary rule is the *General Power Rule* with the *Chain Rule* being secondary as follows!

Let $u = \csc 5x$ and $u' = -5 \csc 5x \cot 5x$, we find

$$\frac{dy}{dx} = -5 \csc 5x \cot 5x (2 \csc 5x)$$

and $\frac{dy}{dx} = -10 \csc^2 5x \cot 5x$.

NOTE: Whenever possible you should ALWAYS combine like terms in your solutions and also write constants first!

Problem 16:

Differentiate $y = 5(\cos x^4)^3$.

In this case, the primary rule is the *Constant Multiple Rule* followed by the *General Power Rule* with an embedded *Chain Rule*.

To differentiate $(\cos x^4)^3$ we'll let $u = \cos x^4$ and $u' = -4x^3 \sin x^4$.

Then the derivative of $(\cos x^4)^3$ is $-4x^3 \sin x^4 [3(\cos x^4)^2]$ using the *General Power Rule*

and $\frac{dy}{dx} = 5\{-4x^3 \sin x^4 [3(\cos x^4)^2]\}$ or

$$\frac{dy}{dx} = -60x^3 \sin x^4 (\cos x^4)^2$$

NOTE: Whenever possible you should ALWAYS combine like terms in your solutions and also write constants first!

Problem 17:

Differentiate $f(x) = (\ln x)^3$.

Using the *General Power Rule* with $u = \ln x$ and $u' = \frac{1}{x}$, we find

$$f'(x) = \tfrac{1}{x}[3(\ln x)^2]$$

and $f'(x) = \frac{3}{x}(\ln x)^2$

NOTE: Whenever possible you should ALWAYS combine like terms in your solutions!

Problem 18:

Differentiate $f(x) = \ln(x^2 + 2)$.

Using the *Chain Rule* with $u = x^2 + 2$ and $u' = 2x$, we find

$$f'(x) = \frac{2x}{x^2 + 2}.$$

Problem 19:

Differentiate $f(x) = \ln\sqrt{x + 1}$.

$$f(x) = \ln\sqrt{x+1} = (x+1)^{1/2}$$

Using the *Chain Rule* with $u = (x+1)^{1/2}$ and $u' = \frac{1}{2}(x+1)^{-1/2}$, we find

$$f'(x) = \frac{1}{\sqrt{x+1}} \cdot \frac{1}{2\sqrt{x+1}}$$

$$f'(x) = \frac{1}{\sqrt{x+1}} \cdot \frac{1}{2\sqrt{x+1}} = \frac{1}{2(x+1)}$$

and $f'(x) = \dfrac{1}{2(x+1)}$

Problem 20:

Differentiate $f(x) = \ln\left[\frac{x(x^2+1)^2}{\sqrt{2x^3-1}}\right]$.

Using the *Chain Rule* to differentiate would be quite cumbersome at this point. Therefore, we will apply the following logarithm properties to rewrite the function into the sum of simpler logarithms.

Product Rule: $\log_b MN = \log_b M + \log_b N$

Quotient Rule: $\log_b \frac{M}{N} = \log_b M - \log_b N$

Power Rule: $\log_b M^p = p \cdot \log_b M$

$$\begin{aligned} f(x) &= \ln[x(x^2+1)^2] - \ln\sqrt{2x^3-1} \\ &= \ln x + \ln(x^2+1)^2 - \ln(2x^3-1)^{1/2} \end{aligned}$$

And finally,

$$f(x) = \ln x + 2\ln(x^2+1) - \tfrac{1}{2}\ln(2x^3-1)$$

Now we will use the *Sum/Difference Rule* to differentiate each term separately. Please note that we have to use the *Chain Rule* to differentiate the second and third term!

$$f'(x) = \frac{1}{x} + \frac{2(2x)}{x^2+1} + \frac{6x^2}{2(2x^3-1)}$$

and $f'(x) = \frac{1}{x} + \frac{4x}{x^2+1} + \frac{3x^2}{2x^3-1}$

Problem 21:

Differentiate $f(x) = e^{2x-1}$.

Using the *Chain Rule* with $u = 2x - 1$ and $u' = 2$, we find

$$f'(x) = 2e^{2x-1}$$

Problem 22:

Differentiate $f(x) = \frac{1}{e^{3/x}}$

First we will write the function as $f(x) = e^{-3/x}$.

Now we will apply the *Chain Rule* with $u = -\frac{3}{x} = -3x^{-1}$ and $u' = 3x^{-2}$.

We find

$$f'(x) = -3x^{-2}e^{-3/x}$$

and $$f'(x) = \frac{-3}{x^2 e^{3/x}}$$

NOTE: Whenever possible you should ALWAYS write your solutions with positive exponents!

Problem 23:

Differentiate $f(x) = \log(\cos x)$.

Using the *Chain Rule* with $u = \cos x$ and $u' = -\sin x$, we find

$$f'(x) = \frac{-\sin x}{\ln 10 \cos x}$$

Problem 24:

Differentiate $f(x) = 2^{3x}$.

Using the *Chain Rule* with $u = 3x$ and $u' = 3$, we find

$$f'(x) = (3\ln 2)(2^{3x})$$