



DETAILED SOLUTIONS AND CONCEPTS - INTRODUCTION TO FRACTIONS

Prepared by Ingrid Stewart, Ph.D., College of Southern Nevada

Please Send Questions and Comments to ingrid.stewart@csn.edu. Thank you!

YOU MUST BE ABLE TO DO THE FOLLOWING PROBLEMS WITHOUT A CALCULATOR!

Aside from decimals, our counting system also lets us write fractional amounts of whole numbers using a vertical bar or a slash. When a fractional amount is written using a vertical bar or slash, we call the number a **fraction**.

All **fractions** have three parts, a **numerator**, a **denominator**, and a **fraction symbol**.

For example, fraction with a **numerator** of **3** and a **denominator** of **4** is written as follows:

$\frac{3}{4}$ or $\frac{3}{4}$ or $3/4$, where the bar (—) and the slash (/) are the **fraction symbols**.

However, in mathematics the bar symbol, that is $\frac{3}{4}$, is the preferred representation.

Please note that a fraction simply indicates division. For example, the fraction $\frac{3}{4}$ means

$3 \div 4$ or $4 \overline{)3}$.

Simple Fractions

Fractions in which the numerator and denominator are whole numbers.

For example, $\frac{2}{5}$.

Complex Fractions

Fractions in which the numerator and denominator are NOT whole numbers.

For example, $\frac{0.78}{9}$, $\frac{1.23}{679.54}$, $\frac{\frac{1}{2}}{\frac{3}{4}}$, $\frac{\frac{1}{3}}{\frac{1}{4} + \frac{1}{5}}$, etc.

Proper Fractions

Fractions in which the numerator is less than the denominator.

For example, $\frac{2}{5}$ (also a simple fraction!).

Improper Fractions

Fractions in which the numerator is greater than the denominator.

for example, $\frac{7}{2}$ (also a simple fraction!).

Compound Fractions or Mixed Numbers

Numbers that have a whole part and a fractional part.

For example, $3\frac{1}{2}$. This actually means $3 + \frac{1}{2}$.

Equivalent Fractions

Fractions that are in the same family of fractions. The first member of the family is a **fraction reduced to lowest terms**. That is, both the numerator and the denominator do NOT have a factor in common other than 1.

For example, $\frac{2}{3}$ is the first member of a family, which includes $\frac{4}{6}$, $\frac{6}{9}$, $\frac{8}{12}$, etc.

Problem 1:

$\frac{3}{4}$ is a fraction reduced to lowest terms. Neither the **3** nor the **4** have a factor in common other than 1. It is considered to be the **FIRST MEMBER** of a **family of fractions**.

Find other family members by multiplying both the numerator and the denominator of the **FIRST MEMBER** by 2, 3, 4, and 5.

$$\frac{3}{4} = \frac{3 \left(\frac{2}{2} \right)}{4 \left(\frac{2}{2} \right)} = \frac{6}{8}$$

$$\frac{3}{4} = \frac{3}{4} \left(\frac{3}{3} \right) = \frac{9}{12}$$

$$\frac{3}{4} = \frac{3}{4} \left(\frac{4}{4} \right) = \frac{12}{16}$$

$$\frac{3}{4} = \frac{3}{4} \left(\frac{5}{5} \right) = \frac{15}{20}$$

Please note that we are using an alternate way of expressing multiplication! Instead of using the multiplication sign, we simply enclose the multiplier in parentheses! See the lecture notes on "Decimals and Whole Numbers."

Problem 2:

Find the value of $\frac{2}{0}$.

The denominator of any fraction cannot have the value **0**. If the denominator of a fraction is **0**, we say that the value of this fraction is **undefined**.

The value of $\frac{2}{0}$ is undefined because the denominator is equal to **0**. Once you encounter such a fraction in a problem ... STOP. You cannot proceed with the problem.

Problem 3:

Find the value of $\frac{0.78}{0}$.

The value of $\frac{0.78}{0}$ is undefined because the denominator is equal to **0**.

Problem 4:

Find the value of $\frac{0}{2}$.

A numerator is allowed to take on the value **0** in a fraction. Any fraction (denominator not equal to **0**) with a numerator equal to **0** has an overall value of **0**.

$\frac{0}{2}$ has a value of **0** because the numerator is equal to **0**.

Problem 5:

Find the value of $\frac{0}{0.78}$.

$\frac{0}{0.78}$ has a value of 0 because the numerator is equal to 0 .

Problem 6:

Find the value of $\frac{0}{0}$.

First we could say that its value is undefined because the denominator is equal to 0 . However, we could also say that its value is 0 because the numerator is 0 . Since there is a disagreement, we must say that the value of $0/0$ is **indeterminate**.

Problem 7:

Write the number 2 as three different fractions using 1, 2, and 3 as the denominator.

You can express any number as a fraction by simply dividing it by 1 or you can express any number as a fraction by simply choosing a numerator and denominator so that the overall value is equal to the number.

$$2 = \frac{2}{1} = \frac{4}{2} = \frac{6}{3} \text{ etc.}$$

Problem 8:

Write the number 0 as three different fractions using 1, 2, and 100,000 as the denominator.

$$0 = \frac{0}{1} = \frac{0}{2} = \frac{0}{100000} \text{ etc.}$$

Problem 9:

Reduce $\frac{2}{2}$ to lowest terms.

The number **1** can take on many forms. Different forms are often needed when working with fractions as you will see later on. When the numerator of a fraction is equivalent to the denominator of a fraction, the value of the fraction is **1**. This only works when the denominator does not equal **0**!!!

$$\frac{2}{2} = 1$$

because the numerator of a fraction is equivalent to the denominator of a fraction.

Problem 10:

Reduce $\frac{1.23}{1.23}$ to lowest terms.

$$\frac{1.23}{1.23} = 1$$

because the numerator of a fraction is equivalent to the denominator of a fraction.

Problem 11:

Reduce $\frac{12}{80}$ to lowest terms.

Let's reduce $\frac{12}{80}$ to lowest terms using the "**Educated Guess**" Method.

When applying some tests for divisibility (see discussion below) we notice that **2** and **4** divide both into the numerator and the denominator. We might as well use the larger number to write

$$\frac{3}{20}$$

and since **3** and **20** have no factor in common except **1**, we have found the FIRST MEMBER of the family of fractions to be

$$\frac{3}{20}$$

, which is a fraction reduced to lowest terms because **3** and **20** have no factor in common except **1**.

Using the **Prime Factorization Method** (see discussion below), we get

$$\frac{12}{80} = \frac{2 \cdot 2 \cdot 3}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 5}$$

We know that there is no change in value when multiplying any number by the number **1**. Therefore, we can reduce the fraction to lowest terms by taking out

$$\frac{2}{2} \cdot \frac{2}{2} = 1 \cdot 1 = 1$$

Therefore,
$$\frac{12}{80} = \frac{3}{2 \cdot 2 \cdot 5} = \frac{3}{20}$$

Tests for Divisibility

A number is divisible by

- 2 if the last digit is an even number (0, 2, 4, 6, 8, etc.)
- 3 if the sum of its digits is divisible by 3
- 4 if the last two digits form a number that is divisible by 4
- 5 if the last digit is 0 or 5
- 6 if the number is divisible by both 2 and 3
- 7 if the division has no remainder
- 8 if the last three digits form a number divisible by 8
- 9 if the sum of its digits is divisible by 9
- 10 if the last digit is 0

Prime Factorization Method

- Write the numerator and denominator as a product of **prime** ** factors and reduce. This is called **prime factorization**.

** A **prime** number is a whole number other than **0** and **1** that is divisible only by itself and **1**. For example, **2, 3, 5, 7, 11, 13, 17**, etc. are prime numbers.

- If a prime factor appears both in the numerator and denominator take them both out because they represent the number **1**.

To illustrate this method, let's write the numerator and denominator of $\frac{12}{80}$ as a product of prime numbers. We start with the smallest prime number after the number **1** and, in turn, proceed to the next higher one until the multiplier equals **1**.

Factor **12** into prime factors.

2 is the smallest prime number and it is a factor of **12**:

$$12 = 2 \cdot 6$$

2 is the smallest prime number and it is a factor of **6**:

$$6 = 2 \cdot 3$$

2 is the smallest prime number and it is not a factor of **3**. As a matter of fact, **3** is a prime number!

$$3 = 3 \cdot 1$$

We are done with the prime factorization process when the multiplier equals 1!

This shows that **12** can be written as a product of prime factors as follows:

$$12 = 2 \cdot 2 \cdot 3$$

Factor **80** into prime factors.

2 is the smallest prime number and it is a factor of **80**:

$$80 = 2 \cdot 40$$

2 is the smallest prime number and it is a factor of **40**:

$$40 = 2 \cdot 20$$

2 is the smallest prime number and it is a factor of **20**:

$$20 = 2 \cdot 10$$

2 is the smallest prime number and it is a factor of **10**:

$$10 = 2 \cdot 5$$

2 is the smallest prime number and it is not a factor of **5**. **3** is the next higher prime number and it is also not a factor of **5**. As a matter of fact, **5** is a prime number!

$$5 = 5 \cdot 1$$

We are done with the prime factorization process when the multiplier equals 1!

This shows that **80** can be written as a product of prime factors as follows:

$$80 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 5$$

Now we can write $\frac{12}{80} = \frac{2 \cdot 2 \cdot 3}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 5}$

Problem 12:

At a concert, 120 of 348 available seats were occupied. Express this finding in fractional form reduced to lowest terms.

$$\frac{120}{348}$$

In fractional form, we have $\frac{120}{348}$. However, this is not reduced to lowest terms.

We will use "educated guessing" and the tests for divisibility to reduce this number to lowest terms! First we notice that the sum of the digits in both the numerator and denominator are divisible by 3 and we get

$$\frac{40}{116}$$

Then, we notice that the last two digits of both the numerator and denominator are divisible by 4 and we get

$$\frac{10}{29}$$

This is the final answer because **10** and **29** have no factor in common except **1**.

We can now restate the earlier finding and say that at this concert $\frac{10}{29}$ of the seats were occupied.

Problem 13:

$$\frac{7}{2}$$

Write the improper fraction $\frac{7}{2}$ as a mixed number.

Sometimes, we need to express an improper fraction as a mixed number. In this case, we simply perform the division and then express the remainder of the division as a fraction!

$$\begin{array}{r} 3 \\ 2 \overline{)7} \\ -6 \\ \hline 1 \end{array}$$

We can see that when we divide **2** into **7** our quotient equals **3** Remainder **1**. **Since we divided by 2, our remainder can be written as $\frac{1}{2}$.**

Therefore, we can write $\frac{7}{2} = 3\frac{1}{2}$

NOTE:
means

$$3 + \frac{1}{2} \text{!!!!!!!!!!!!}$$

Problem 14:

Write the improper fraction $\frac{235}{17}$ as a mixed number.

$$\begin{array}{r} 13 \\ 17 \overline{)235} \\ \underline{-17} \\ 65 \\ \underline{-51} \\ 14 \end{array}$$

We can see that when we divide 17 into 235 our quotient equals 13 Remainder 14.

Since we divided by 17, our remainder can be written as $\frac{14}{17}$.

Therefore, we can write $\frac{235}{17} = 13\frac{14}{17}$

NOTE:
means

$$13 + \frac{14}{17} \text{!!!!!!!!!!!!}$$

Problem 15:

Write the mixed number $3\frac{1}{2}$ as an improper fraction.

Earlier we converted improper fractions to mixed numbers. However, sometimes we might want to convert mixed numbers to decimals. This is actually quite easy.

We simply multiply the whole number by the denominator of the fractional part. To this product we then add the numerator of the fractional part. The denominator stays the same throughout.

$$3\frac{1}{2} = \frac{3 \cdot 2 + 1}{2} = \frac{6 + 1}{2} = \frac{7}{2}$$

Problem 16:

Write the mixed number $21\frac{5}{6}$ as an improper fraction.

$$21\frac{5}{6} = \frac{21 \cdot 6 + 5}{6} = \frac{126 + 5}{6} = \frac{131}{6}$$

Problem 17:

Write the fraction $\frac{3}{4}$ as a decimal without using the calculator.

The fraction $\frac{3}{4}$ means $3 \div 4$ or $4 \overline{)3}$ and when carrying out this division, we get a decimal number as follows.

$$\begin{array}{r} 0.75 \\ 4 \overline{) 3.0000} \\ \underline{-28} \\ 20 \\ \underline{-20} \\ 0 \end{array}$$

This shows that the fraction $\frac{3}{4}$ is equivalent to the decimal number **0.75**. **This decimal is called a terminating decimal because it has a finite number of decimal places.**

Problem 18:

Write the fraction $\frac{1}{10}$ as a decimal without using the calculator.

This fraction has a denominator that is a **power of 10**. Any fraction whose denominator is 10, 100, 1000, etc can be written as a decimal without making any calculations by following the rule below:

- **Use the denominator to find the number of decimal places.**

A denominator of 10 (one zero) means one (tenths) decimal place.

A denominator of 100 (two zeros) means two (hundredths) decimal places.

A denominator of 1000 (three zeros) means three (thousandths) decimal places.

etc.

- **Place the numerator so that the last digit is in the rightmost decimal place.**
- **Fill in blank spaces with zeros, if necessary.**

Therefore, in the case of $\frac{1}{10}$, the denominator of 10 means ONE decimal place and

$$\frac{1}{10} = 0.1$$

Problem 19:

Write the fraction $\frac{3}{10}$ as a decimal without using the calculator.

Since we know that $\frac{1}{10} = 0.1$ and $\frac{3}{10} = 3\left(\frac{1}{10}\right)$, we can state $\frac{3}{10} = 0.3$.

Problem 20:

as Write the fraction $\frac{1}{100}$ a decimal without using the calculator.

In the case of $\frac{1}{100}$, the denominator of 100 means TWO decimal places and

$$\frac{1}{100} = 0.01$$

NOTE: We had to fill one blank space with a zero.

Problem 21:

Write the fraction $\frac{67}{100}$ as a decimal without using the calculator.

Since we know that $\frac{1}{100} = 0.01$ and $\frac{67}{100} = 67 \left(\frac{1}{100} \right)$, we can state $\frac{67}{100} = 0.67$.

Problem 22:

Write the fraction $\frac{1}{1000}$ as a decimal without using the calculator.

, In the case of $\frac{1}{1000}$ the denominator of 1000 means THREE decimal places
and $\frac{1}{1000} = 0.001$.

NOTE: We had to fill two blank spaces with a zero.

Problem 23:

Write the fraction $\frac{81}{1000}$ as a decimal without using the calculator.

Since we know that $\frac{1}{1000} = 0.001$ and $\frac{81}{1000} = 81 \left(\frac{1}{1000} \right)$, we can state
 $\frac{81}{1000} = 0.081$.

NOTE: We had to fill one blank space with a zero.

Problem 24:

Write the fraction $\frac{213}{1000}$ as a decimal without using the calculator.

Since we know that $\frac{1}{1000} = 0.001$ and $\frac{213}{1000} = 213 \left(\frac{1}{1000} \right)$, we can state
 $\frac{213}{1000} = 0.213$.

Problem 25:

Write the fraction $\frac{2}{3}$ as a decimal without using the calculator.

In the case $\frac{2}{3} = 3 \overline{)2}$ of , using long division we find that its decimal equivalent is $0.666666666\dots$, with the **6** repeating endlessly.

This decimal is called a non-terminating, repeating decimal.

A calculator would eventually round one of the 6's to **7**. Often $\frac{2}{3}$ is therefore expressed as **0.67**, which is an approximation.

EXACTLY, we express $\frac{2}{3}$ as the decimal number $0.\overline{6}$, pronounced "0 point 6 repeating" as indicated by the horizontal bar over the **6**.

Problem 26:

Write the fraction $\frac{9}{11}$ as a decimal without using the calculator.

In the case $\frac{9}{11} = 11 \overline{)9}$ of , using long division we find that its decimal equivalent is $0.81818181\dots$, with the **81** repeating endlessly. This is another **non-terminating, repeating decimal**.

EXACTLY, we express $\frac{9}{11}$ as the decimal number $0.\overline{81}$, pronounced "0 point 81 repeating" as indicated by the horizontal bar over the **81**.

Problem 27:

Write the mixed number $15\frac{7}{8}$ as a number containing a whole and a decimal part.

We only need to find the decimal equivalent of the fractional part. That is, $\frac{7}{8} = 0.875$ using long division.

Then $15\frac{7}{8} = 15.875$.

Problem 28:

Write **0.7** as a fraction.

Earlier we converted fractions to decimals. However, sometimes we might want to convert decimals to fractions. This is actually quite easy using the following rule.

- **Count the decimal places.**
- **Convert each decimal place into a zero.**
- **Add the number 1 on the left. This is the denominator of the fractional equivalent.**
- **The numerator of the fractional equivalent consists of the numbers to the right of the decimal place without leading zeros.**

0.7 has one decimal place! This is converted to **0** and **1** is added to its left to make **10**. This is the denominator of our fraction. The numerator is **7**.

$$0.7 = \frac{7}{10}$$

Problem 29:

Write **0.39** as a fraction.

0.39 has two decimal places! This is converted to **0's** and **1** is added to their left to make **100**. This is the denominator of our fraction. The numerator is **39**.

$$0.39 = \frac{39}{100}$$

Problem 30:

Write **0.065** as a fraction.

We count three decimal places, convert them to **0**, and add a **1** to the left to make **1000**. This is the denominator of our fraction. The numerator is **65**. **We dropped the leading zero!**

$$0.065 = \frac{65}{1000}$$

However, this fraction is not reduced to lowest terms. Actually, we can do the following:

$$\begin{aligned} \frac{65}{1000} &= \frac{5 \cdot 13}{5 \cdot 200} \\ &= \frac{13}{200} \quad \text{having reduced by } \frac{5}{5} = 1 \end{aligned}$$

Problem 31:

Write **0.0001** as a fraction.

We count four decimal places, convert them to **0**, and add a **1** to the left to make **10,000**. This is the denominator of our fraction. The numerator is **1**. **We dropped the leading zeros!**

$$0.0001 = \frac{1}{10,000}$$

Problem 32:

Write **0.01** as a fraction.

We count two decimal places, convert them to **0**, and add a **1** to the left to make **100**. This is the denominator of our fraction. The numerator is **1**. **We dropped the leading zero!**

$$0.01 = \frac{1}{100}$$

Problem 33:

Write the mixed number **45.153** as a mixed number containing a fraction.

We only need to find the fractional equivalent of the decimal digits.

That is,
$$0.153 = \frac{153}{1,000}$$

Then
$$45.153 = 45 \frac{153}{1,000}$$

Problem 34:

Multiplying the number **135,000** by **0.001** is equivalent to dividing **135,000** by what number?

We count three decimal places, convert them to **0**, and add a **1** to the left to make **1,000**. This is the denominator of our fraction. The numerator is **1**. **We dropped the leading zeros!**

$$0.001 = \frac{1}{1,000}$$

We have found that multiplying **135,000** by **0.001** is equivalent to dividing it by **1,000**.

Problem 35:

Multiplying some number by **0.1** is equivalent to dividing this number by what number?

We count one decimal place, convert it to **0**, and add a **1** to the left to make **10**. This is the denominator of our fraction. The numerator is **1**.

$$0.1 = \frac{1}{10}$$

We have found that multiplying some number by **0.1** is equivalent to dividing this number by **10**.

Problem 36:

Multiplying some number by **0.00001** is equivalent to dividing this number by what number?

We count five decimal places, convert them to **0**, and add a **1** to the left to make **100,000**. This is the denominator of our fraction. The numerator is **1**. **We dropped the leading zeros!**

$$0.00001 = \frac{1}{100,000}$$

We have found that multiplying some number by **0.00001** is equivalent to dividing this number by **100,000**.