DETAILED SOLUTIONS AND CONCEPTS - RATIONAL EXPRESSIONS AND EQUATIONS

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PLEASE NOTE THAT YOU CANNOT USE A CALCULATOR ON THE ACCUPLACER - ELEMENTARY ALGEBRA TEST! YOU MUST BE ABLE TO DO THE FOLLOWING PROBLEMS WITHOUT A CALCULATOR!

Simplifying Rational Expressions (Reducing to Lowest Term)

- Factor both the numerator and denominator relative to the integers. Hint: Try to find the greatest common factor first!
- Reduce factors common to both the numerator and denominator.

Adding and Subtracting Rational Expressions

- Factor the numerator and denominator relative to the integers.
- Cross-cancel as much as possible.
- Find the **Least Common Denominator (LCD)** by factoring each denominator into **prime factors** and forming a product of all distinct prime factors. Use the factor to highest power if two or more factors are not distinct.
- Change each fraction to an equivalent fraction using the LCD.
- Add or subtract numerators keeping the same denominator.
- Reduce, if possible.

The **Least Common Denominator (LCD)** is the smallest number that is evenly divisible (no remainder) by each of the denominators of the fractions being added. If the denominators are all prime numbers, the LCD is the product of the denominators. If one or more of the denominators is NOT a prime number, the LCD is a product using the largest listing of of each prime number taking into account ALL denominators.

the LCD is the product found by using certain prime number factors of each denominator.

A **prime** number is a whole number other than ***0*** and ***1*** that is divisible only by itself and ***1.*** For example, ***2***, ***3***, ***5***, ***7***, ***11***, ***13***, ***17***, etc. are prime numbers. In the case of polynomial expressions, they are called "prime" if they cannot be factored relative to the integers.

Multiplying and Dividing Rational Expressions

Multiplication Rule:

- Factor the numerator and denominator relative to the integers.
- Cross-cancel as much as possible.
- Multiply the numerators AND multiply the denominators.

Division Rule:

- Convert the division to an equivalent multiplication problem using the **reciprocal **** of the divisor as the multiplier.
- Then multiply according to the **rule for multiplying rational expressions.**

  ****** Interchanging the numerator and denominator of a fraction results in a fraction that is called the **reciprocal** of the original fraction. When a number is multiplied by its reciprocal, the product equals ***1***.

Finding Solutions of Rational Equations

- Find the **Least Common Denominator (LCD)**.
- Multiply both sides of the equation by the LCD, reduce, and solve.
- Check the solutions in the original equation, rejecting any that produce ***0*** or an **imaginary number** in the denominator.

Problem 1:

Simplify $\frac{a+b}{4a^2+4ab}$.

Please note that you are asked to "simplify". This means that you have to reduce the rational expression to lowest term.

The numerator cannot be factored relative to the integers. Let's factor out the greatest common factor in the denominator.

$$\frac{a+b}{4a(a+b)}$$

Since we know that $\frac{a+b}{a+b}=1$,

we can further reduce to get $\frac{1}{4a}$, which is the final reduced form.

Problem 2:

Simplify $\dfrac{x^2-8x+16}{x^2-16}$.

Let's factor the numerator and denominator relative to the integers

$$\frac{(x-4)(x-4)}{(x-4)(x+4)}$$

Since we know that $\dfrac{x-4}{x-4}=1$,

we can further reduce to get $\dfrac{x-4}{x+4}$, which is the final reduced form.

Problem 3:

Simplify $\dfrac{6x^2-24}{3x+6}$.

Let's factor out the greatest common factor in the numerator and the denominator.

$$\frac{6(x^2-4)}{3(x+2)}$$

Now, we know that $\dfrac{6}{3}=2$ and we can also further factor the denominator to get

$$\frac{2(x-2)(x+2)}{x+2}$$

Since we know that $\dfrac{x+2}{x+2}=1$,

we can further reduce to get $2(x-2)$ or $2x-4$, which is the final reduced form.

Problem 4:

Simplify $\dfrac{x-y}{y-x}$.

This is somewhat "tricky." Notice that we have an x and a y in the numerator and the denominator. But their signs are opposite.

Whenever this happens, simply factor a **-1** out of the numerator OR the denominator.

Let's see what happens if we factor a **-1** out of the denominator:

$$\frac{x-y}{-(-y+x)}=\frac{x-y}{-(x-y)}=-1$$

We found the final reduced form to be a constant, namely the number ***-1***.

Problem 5:

Simplify $\frac{r}{r+r^2}$.

The numerator cannot be factored relative to the integers. Let's factor out the greatest common factor in the denominator.

$$\frac{r}{r(1+r)}$$

Since we know that $\frac{r}{r}=1$,

we can further reduce to get $\frac{1}{1+r}$, which is the final reduced form.

Problem 6:

Simplify $\frac{(x+2)(x-3)^2}{x^4-13x^2+36}$.

Let's factor the numerator and denominator. Please note that we have a polynomial expression in the denominator that behaves "like a quadratic"!

$$\frac{(x+2)(x-3)(x-3)}{(x^2-9)(x^2-4)}$$

We also notice that the two factors in the denominator are *Differences of Squares* which can be factored as follows:

$$\frac{(x+2)(x-3)(x-3)}{(x-3)(x+3)(x-2)(x+2)}$$

we can now reduce to get $\dfrac{x-3}{(x+3)(x-2)}$, which can also be written as $\dfrac{x-3}{x^2+x-6}$.

Problem 7:

Add $\dfrac{4}{x+2}+\dfrac{3}{x-2}$.

The numerators and denominators cannot be factored relative to the integers.

Let's find the **Least Common Denominator**.

Since both denominators are prime, the LCD is the product found by multiplying both denominators.

Therefore, the LCD is $(x+2)(x-2)$!!!

Next, we change each fraction to an equivalent fraction using the LCD.

$$\frac{4(x-2)}{(x+2)(x-2)}+\frac{3(x+2)}{(x+2)(x-2)}$$

$$\frac{4x-8}{(x+2)(x-2)}+\frac{3x+6}{(x+2)(x-2)}$$

then we add the numerators keeping the same denominator

$$\frac{4x-8+3x+6}{(x+2)(x-2)}$$

and the final sum is $\dfrac{7x-2}{(x+2)(x-2)}$ or $\dfrac{7x-2}{x^2-4}$.

Problem 8:

Subtract $\dfrac{x}{x-1}-\dfrac{6}{x+3}$.

The numerators and denominators cannot be factored relative to the integers.

Let's find the **Least Common Denominator**.

Since both denominators are prime, the LCD is the product found by multiplying both denominators.

Therefore, the LCD is $(x-1)(x+3)$!!!

Next, we change each fraction to an equivalent fraction using the LCD.

$$\frac{x(x+3)}{(x-1)(x+3)} - \frac{6(x-1)}{(x-1)(x+3)}$$

and $\frac{x^2+3x}{(x-1)(x+3)} - \frac{6x-6}{(x-1)(x+3)}$ Please note that the minus sign is in front of the ENTIRE numerator ***6x - 6*** !

Therefore, we have to keep the minus sign is in front of the ENTIRE numerator as follows:

$$\frac{x^2+3x-(6x-6)}{(x-1)(x+3)}$$

and $\frac{x^2+3x-6x+6}{(x-1)(x+3)}$

The final sum is $\frac{x^2-3x+6}{(x-1)(x+3)}$ or $\frac{x^2-3x+6}{x^2+2x-3}$.

Problem 9:

Subtract $\frac{3}{4a} - \frac{5}{a^2}$.

The numerators cannot be factored relative to the integers.

Let's find find the **Least Common Denominator**.

Factor all denominators into prime factors as follows:

$4a$: This expression is not "prime" because we can factor it as $2 \cdot 2 \cdot a$.

a^2: This expression is not "prime" because can factor it as $a \cdot a$.

Since the denominators are NOT prime, the LCD is the product found by creating a product using the largest listing of of each prime expression taking into account ALL denominators.

In our case, the number 2 is listed twice and the expression a is listed twice. Therefore, the LCD is $2 \cdot 2 \cdot a \cdot a = 4a^2$!!!

Next, we change each fraction to an equivalent fraction using the LCD.

$\frac{3a}{4a^2} - \frac{5(4)}{4a^2}$ and $\frac{3a - 20}{4a^2}$

Problem 10:

Add $\frac{x}{x+3} + \frac{2x+4}{(x+2)(x+3)}$.

One numerator can be factored relative to the integers, but the denominators are already factored.

$$\frac{x}{x+3} + \frac{2(x+2)}{(x+2)(x+3)}$$

Now we can reduce the second term as follows:

$$\frac{x}{x+3} + \frac{2}{(x+3)}$$

and we can immediately state the sum to be $\frac{x+2}{x+3}$.

Problem 11:

Subtract $\frac{5}{x+4} - \frac{4}{x}$.

The numerators and denominators cannot be factored relative to the integers.

Let's find the **Least Common Denominator**.

Since both denominators are prime, the LCD is the product found by multiplying both denominators.

Therefore, the LCD is $x(x+4)$!!!

Next, we change each fraction to an equivalent fraction using the LCD.

$$\frac{5x}{x(x+4)} - \frac{4(x+4)}{x(x+4)}$$

and $\frac{5x}{x(x+4)} - \frac{4x+16}{x(x+4)}$ Please note that the minus sign is in front of the ENTIRE numerator ***4x + 16*** !

Therefore, we have to keep the minus sign is in front of the ENTIRE numerator as follows:

$$\frac{5x-(4x+16)}{x(x+4)}$$

and $\frac{5x-4x-16}{x(x+4)}$

The final sum is $\frac{x-16}{x(x+4)}$ or $\frac{x-16}{x^2+4x}$.

Problem 12:

Multiply $\frac{x+2}{x} \cdot \frac{x^2+3x}{x^2+5x+6}$.

First we are going to factor the numerators and the denominators.

$$\frac{x+2}{x} \cdot \frac{x(x+3)}{(x+2)(x+3)}$$

Next we will cross-cancel as follows:

$$\frac{\cancel{x+2}}{\cancel{x}} \cdot \frac{\cancel{x}(\cancel{x+3})}{(\cancel{x+2})(\cancel{x+3})} = 1$$

Problem 13:

Divide $\frac{x+2}{x^2-4} \div \frac{x^2+x-2}{x+2}$.

First we are going to factor the numerators and the denominators.

$$\frac{x+2}{(x-2)(x+2)} \div \frac{(x-1)(x+2)}{x+2}$$

Next we will cross-cancel within each term to get:

$$\frac{1}{x-2} \div \frac{x-1}{1}$$

Then, we convert the division to an equivalent multiplication problem using the **reciprocal** of the divisor as the multiplier.

$$\frac{1}{x-2} \cdot \frac{1}{x-1}$$

and we get $\frac{1}{(x-2)(x-1)}$ or $\frac{1}{x^2-3x+2}$.

Problem 14:

Write the compound fraction $\frac{1}{\frac{1}{x}-\frac{1}{y}}$ as a simple fraction.

In this case, we first find the Least Common Denominator taking into account all of the denominators. In our case, ***x*** and ***y***.

Since both denominators are prime, the LCD is the product found by multiplying both denominators.

Therefore, the LCD is ***xy*** !!!

Then, we multiply both the numerator and denominator of the compound fraction by the LCD.

$$\frac{1}{\frac{1}{x}-\frac{1}{y}} \cdot \frac{xy}{xy}$$

Next, we must distribute the numerator of the LCD to the numerator of the compound fraction and, furthermore, we must distribute the denominator of the LCD to ALL terms of the denominator of the compound fraction.

$$\frac{xy}{\frac{xy}{x}-\frac{xy}{y}}$$

finally we reduce in the denominator to get the simple fraction $\frac{xy}{y-x}$.

Problem 15:

Solve $\frac{5}{x-2}-\frac{17-x}{2x-4}=0$.

First, we find the Least Common Denominator. We need to factor all denominators into prime factors as follows:

x - 2: This expression is called "prime" because it is not factorable relative to the integers.

2x - 4: This expression is not "prime" because we can factor out a ***2*** and write ***2(x - 2)***.

$$\frac{5}{x-2}-\frac{17-x}{2(x-2)}=0$$

Since one denominator is NOT prime, the LCD is the product found by creating a product using the largest listing of of each prime expression taking into account ALL denominators.

In our case, the number ***2*** is listed once and the expression ***x - 2*** is listed once. Therefore, the LCD is ***2(x - 2)*** !!!

Next, we multiply both sides of the equation by the LCD

$$2(x-2)\left[\frac{5}{x-2}-\frac{17-x}{2(x-2)}\right]=2(x-2)(0)$$

then we distribute the LCD to **ALL** terms on each side

$$\frac{2(x-2)(5)}{x-2}-\frac{2(x-2)(17-x)}{2(x-2)}=2(x-2)(0)$$

Please note that $\frac{x-2}{x-2}=1$ and $\frac{2}{2}=1$ and $2(x-2)(0)=0$!!!!!!!!!!!!!!!!!!!!!!!!

and finally we reduce each term and solve

$$2(5)-(17-x)=0$$

$$10-17+x=0$$

$$x=7$$

Any time we solve a rational equation we **MUST** check the solutions in the original equation, rejecting any that produce ***0*** or an **imaginary number** in the denominator.

In our case, we must replace the variable with ***7*** in the original equation to make sure that the value to the left of the equal sign becomes equal to the one on the right, which is ***0***.

$$\frac{5}{7-2}-\frac{17-7}{2(7)-4}=1-1=0$$

Since the value of the left side equals 0, which is also the value of the right side, we can say that the solution is indeed *7*.

Problem 16:

Solve $\frac{5}{x+6}-\frac{3}{x}=0$.

First, we find the Least Common Denominator. We need to factor all denominators into prime factors as follows:

x + 6: This expression is called "prime" because it is not factorable relative to the integers.

x: This expression is called "prime" because it is not factorable relative to the integers.

Since both expressions are prime, the LCD is found by multiplying them. Therefore, the LCD is ***x(x + 6)*** !!!

Multiplying both sides by the LCD we get

$$x(x+6)\left(\frac{5}{x+6}-\frac{3}{x}\right)=x(x+6)(0)$$

then we distribute the LCD to **ALL** terms on each side

$$\frac{x(x+6)(5)}{x+6}-\frac{x(x+6)(3)}{x}=x(x+6)(0)$$

Please note that $\frac{x+6}{x+6}=1$ and $\frac{x}{x}=1$ and $x(x+6)(0)=0$!!!!!!!!!!!!!!!!!!!!!!!!!

and finally we reduce each term and solve

$$5x-3(x+6)=0$$
$$5x-3x-18=0$$
$$2x=18$$
$$x=9$$

Again, we must check the solutions in the original equation, rejecting any that produce ***0*** or an **imaginary number** in the denominator.

In this case, we must replace the variable with ***9*** in the original equation to make sure that the value to the left of the equal sign becomes equal to the one on the right, which is ***0***.

$$\frac{5}{9+6}-\frac{3}{9}=\frac{1}{3}-\frac{1}{3}=0$$

Since the value of the left side equals 0, which is also the value of the right side, we can say that the solution is indeed *9*.

Problem 17:

Solve $\frac{x^2+x-6}{x^2-8x+12}=0$.

Since we only have one rational expression, we simply multiplying both sides by the denominator

$$(x^2-8x+12)\left(\frac{x^2+x-6}{x^2-8x+12}\right)=0(x^2-8x+12)$$

and when we multiply the left side, we get

$$x^2+x-6=0$$

Please note that $\frac{x^2-8x+12}{x^2-8x+12}=1$!!!!!!!!!!!!!!!!!!!!!!!!

and then we reduce to get

$$x^2+x-6=0$$

The trinomial on the right can be factored as follows

$$(x-2)(x+3)=0$$

and then we can use the *Zero Product Principle* to find the solutions

$x-2=0$ and $x+3=0$

so that $x=2$ and $x=-3$

Again we must check the solutions in the original equation, rejecting any that produce ***0*** or an imaginary number in the denominator.

First, we must replace the variable with ***-3*** in the original equation to make sure that the value to the left of the equal sign becomes equal to the one on the right, which is ***0***.

For $x = -3$

$$\frac{(-3)^2 + (-3) - 6}{(-3)^2 - 8(-3) + 12} = \frac{0}{45} = 0$$

Since the value of the left side equals 0, which is also the value of the right side, we can say that the solution is indeed *-3*.

Next, we must replace the variable with ***2*** in the original equation to make sure that the value to the left of the equal sign becomes equal to the one on the right, which is ***0***.

For $x = 2$

$$\frac{(2)^2 + 2 - 6}{(2)^2 - 8(2) + 12} = \frac{0}{0} = undefined$$

Since the value of the left side is undefined, we can say that *2* is NOT a solution.

Note that any time a ***0*** appears in the denominator the expression becomes undefined !

We find that this rational equation only has ONE solution, namely *-3*.

Problem 18:

Solve $\frac{2}{x+1} + \frac{3}{3x+3} = \frac{5}{3}$.

First, we find the Least Common Denominator.

Factor all denominators into prime factors as follows:

3: This number is "prime".

x + 1: This expression is called "prime" because it is not factorable relative to the integers.

3x + 3: This expression is not "prime" because we can factor out a ***3*** and write ***3(x + 3)***.

$$\frac{2}{x+1} + \frac{3}{3(x+1)} = \frac{5}{3}$$

Since one denominator is NOT prime, the LCD is the product found by creating a product using the largest listing of of each prime expression taking into account ALL denominators.

In our case, the number ***3*** is listed once and the expression ***x + 1*** is listed once. Therefore, the LCD is ***3(x + 1)*** !!!

Next, we multiply both sides of the equation by the LCD

$$3(x+1)\left[\frac{2}{x+1}+\frac{3}{3(x+1)}\right]=3(x+1)\left(\frac{5}{3}\right)$$

then we distribute the LCD to **ALL** terms on each side and reduce

$$\frac{2[3(x+1)]}{x+1}+\frac{3[3(x+1)]}{3(x+1)}=5(x+1)$$

and $6+3=5x+5$

and finally we reduce each term and solve

$$5x=4$$

and $x=\frac{4}{5}$

Any time we solve a rational equation we **MUST** check the solutions in the original equation, rejecting any that produce ***0*** or an **imaginary number** in the denominator.

In our case, we must replace the variable with $\frac{4}{5}$ in the original equation to make sure that the value to the left of the equal sign becomes equal to the one on the right, which is $\frac{5}{3}$.

$$\frac{2}{\frac{4}{5}+1}+\frac{3}{3(\frac{4}{5})+3}$$

and adding in the denominators, we get $\frac{2}{\frac{4}{5}+\frac{5}{5}}+\frac{3}{\frac{12}{5}+\frac{15}{5}}$

this results in the following complex fraction $\frac{2}{\frac{9}{5}}+\frac{3}{\frac{27}{5}}$, which can be simplified as follows:

$$
\begin{aligned}
2\left(\frac{5}{9}\right) + 3\left(\frac{5}{27}\right) &= \frac{10}{9} + \frac{15}{27} \\
&= \frac{30}{27} + \frac{15}{27} \\
&= \frac{45}{27} \\
&= \frac{5}{3}
\end{aligned}
$$

Since the value of the left side equals $\frac{5}{3}$, which is also the value of the right side, we can say that the solution is indeed $\frac{4}{5}$.