DETAILED SOLUTIONS AND CONCEPTS - QUADRATIC EQUATIONS

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PLEASE NOTE THAT YOU CANNOT USE A CALCULATOR ON THE ACCUPLACER - ELEMENTARY ALGEBRA TEST! YOU MUST BE ABLE TO DO THE FOLLOWING PROBLEMS WITHOUT A CALCULATOR!

Quadratic Equations

A quadratic equation in x is an equation that can be written in the standard form $ax^2 + bx + c = 0$, where a, b, and c are real numbers, but $a \neq 0$.

Solving Quadratic Equations

A. Factoring Method - Some, but not all quadratic equations can be solved by factoring.

- Write the quadratic equation in standard form. That is, using *Properties of Equality*, bring all terms to one side of the equation so that the other side is equal to 0. Combine like terms, if possible.
- Factor relative to the integers.
- Use the *Zero Product Principle* to solve. It states that if a product is equal to zero, then at least one of the factors is equal to zero.

Problem 1:

Solve $15x^2 - 5x = 0$ using the *Factoring Method*. Find any real and imaginary solutions.

The quadratic equation is already in standard form so that we can factor as follows:

$$5x(3x - 1) = 0$$

By the *Zero Product Principle* we can the say

$5x = 0$ or $3x - 1 = 0$

Therefore, $x = 0$ or $x = \frac{1}{3}$.

Check:

Please note that if we check these solutions in the original equation, we find the following:

Given $15x^2 - 5x = 0$ and using $x = 0$, is $15(0)^2 - 5(0)$ equal to 0? Yes! Therefore the solution is correct!

Given $15x^2 - 5x = 0$ and using $x = \frac{1}{3}$, is $15(\frac{1}{3})^2 - 5(\frac{1}{3})$ equal to 0? Yes, because $\frac{15}{9} - \frac{5}{3} = \frac{15}{9} - \frac{15}{9}$! Therefore the solution is correct!

Problem 2:

Solve $x^2 + 5x = -6$ using the *Factoring Method*. Find any real and imaginary solutions.

Rewriting the equation in standard form yields $x^2 + 5x + 6 = 0$.

We notice that we have a trinomial, which we can factor as follows:

$$(x + 3)(x + 2) = 0$$

By the *Zero Product Principle* we can the say

$x + 3 = 0$ or $x + 2 = 0$

Therefore, $x = -3$ or $x = -2$.

Check:

Please note that if we check these solutions in the original equation, we find the following:

Given $x^2 + 5x = -6$ and using $x = -3$, is $(-3)^2 + 5(-3)$ equal to -6? Yes! Therefore the solution is correct!

Given $x^2 + 5x = -6$ and using $x = -2$, is $(-2)^2 + 5(-2)$ equal to -6? Yes! Therefore the solution is correct!

Problem 3:

Solve $x^2 - 4x + 4 = 0$ using the *Factoring Method*. Find any real and imaginary solutions.

We notice that we have a trinomial, which we can factor as follows:

$$(x - 2)(x - 2) = 0$$

By the *Zero Product Principle* we can the say

$x - 2 = 0$ or $x - 2 = 0$

Therefore, $x = 2$ in both cases. Sometimes the 2 is considered to be a **double root**.

B. Square Root Method - All quadratic equations can be solved by this method which uses the *Square Root Property*. However, we might want to avoid this method for some quadratic equations because it can get very cumbersome.

- Isolate the squared term on one side of the equation. Be sure its coefficient is a ***positive 1*** !!!!!!
- Apply the *Square Root Property*.
- If necessary, further isolate the variable.

The Square Root Property
If u is an algebraic expression containing a variable and d is a constant, then
$u^2 = d$ has exactly two solutions, namely
$u = \sqrt{d}$ and $u = -\sqrt{d}$ or simply $u = \pm\sqrt{d}$.

Problem 4:

Solve $x^2 - 16 = 0$ using the *Square Root Method*. Find any real and imaginary solutions.

Isolate the squared term

$x^2 = 16$

and use the *Square Root Property*

$x = \pm\sqrt{16}$

then $x = 4$ or $x = -4$.

Problem 5:

Solve $x^2 - 5 = 0$ using the *Square Root Method*. Find any real and imaginary solutions.

Isolate the squared term

$x^2 = 5$

and use the *Square Root Property*

$x = \pm\sqrt{5}$

Therefore, $x = \sqrt{5}$ or $x = -\sqrt{5}$.

Problem 6:

Solve $(x - 2)^2 + 8 = 0$ using the *Square Root Method*. Find any real and imaginary solutions.

Isolate the squared term

$$(x - 2)^2 = -8$$

use the *Square Root Property*

$$x - 2 = \pm\sqrt{-8}$$

Please observe that we found an imaginary number. This means we have an imaginary solution.

We can further isolate the variable

$$x = 2 \pm \sqrt{-8}$$

and lastly simplify the square root as follows:

$$x = 2 \pm 2i\sqrt{2}$$

Therefore, $x = 2 + 2i\sqrt{2}$ or $x = 2 - 2i\sqrt{2}$.

C. Quadratic Formula Method - This method can be used to solve any type of quadratic equation. The *Quadratic Formula* was derived by first applying the *Square Completion Method* to the general form of the quadratic equation $ax^2 + bx + c = 0$ and then using the *Square Root Method* to solve for x.

The Quadratic Formula

The solutions of a quadratic equation in general form $ax^2 + bx + c = 0$ with $a \neq 0$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- Write the quadratic equation in standard form. That is, using properties of equality, bring all terms to one side of the equation so that the other side is equal to 0. Combine like terms, if possible.
- a is the coefficient of x^2, b is the coefficient of x, and c is the constant.

Note: b and/or c can be zero or a negative number!!! a cannot be equal to zero, but it can be negative.

- Substitute the values of ***a***, ***b***, and ***c*** into the quadratic formula. Be sure to note that a negative value for ***b*** in the quadratic equation does not replace the negative sign in front of ***b*** in the formula.
- Perform all arithmetic operations to find simplified solutions, such as adding like terms, simplifying square roots, and reducing fractions.

Problem 7:

Solve $x^2 - 10x + 20 = 0$ using the *Quadratic Formula Method*. Find any real and imaginary solutions.

For the *Quadratic Formula*, we need the the values $a = 1$, $b = -10$, and $c = 20$ from the given equation. Then,

$$x = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(20)}}{2(1)}$$

Next, we combine like terms and simplify the radical

$$x = \frac{10 \pm \sqrt{20}}{2} = \frac{10 \pm 2\sqrt{5}}{2}$$

and finally, we reduce the answer to lowest term

$$x = \frac{2(5 \pm \sqrt{5})}{2} = 5 \pm \sqrt{5}$$

Therefore, $x = 5 + \sqrt{5}$ or $x = 5 - \sqrt{5}$.

Problem 8:

Solve $4x^2 - 8x + 11 = 0$ using the *Quadratic Formula Method*. Find any real and imaginary solutions.

In this case, $a = 4 \quad b = -8 \quad c = 11$

$$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(4)(11)}}{2(4)}$$

$$x = \frac{8 \pm \sqrt{64 - 176}}{8}$$

$$x = \frac{8 \pm \sqrt{-112}}{8}$$

Please observe that we found an imaginary number. This means we have an imaginary solution.

We could further simplify to get

$$x = \frac{8 \pm 4i\sqrt{7}}{8}$$

and finally, we will reduce the fraction to find

$$x = \frac{2 \pm i\sqrt{7}}{2}$$

Therefore, $x = \frac{2 + i\sqrt{7}}{2}$ or $x = \frac{2 - i\sqrt{7}}{2}$.

Please note that we could NOT have solved this quadratic equation by the *Factoring Method*.

Problem 9:

Solve $\boldsymbol{x^2 - 9 = 0}$ using the *Quadratic Formula*, *Factoring*, and the *Square Root Property*. Find any real and imaginary solutions.

a. Using the Quadratic Formula

In this case, $a = 1$, $b = 0$, and $c = -9$

$$x = \frac{-0 \pm \sqrt{0^2 - 4(1)(-9)}}{2(1)}$$

$$x = \frac{\pm\sqrt{36}}{2}$$

$$x = \pm 3$$

Therefore, $x = -3$ or $x = 3$.

Please note that we could have solved this quadratic equation also by using the *Factoring* or *Square Root Methods*.

b. Using Factoring and the Zero Product Principle

Solve $\boldsymbol{x^2 - 9 = 0}$.

Notice that the are dealing with a *Difference of Squares*
$x^2 - a^2 = (x - a)(x + a)$, where $a = 3$.

Therefore, we can factor as follows:

$$(x - 3)(x + 3) = 0$$

Therefore, $x = 3$ or $x = -3$.

Check:

Please note that if we check these solutions in the original equation, we find the following:

Given $x^2 - 9 = 0$ and using $x = 3$, is $(3)^2 - 9$ equal to 0? Yes! Therefore the solution is correct!

Given $x^2 - 9 = 0$ and using $x = -3$, is $(-3)^2 - 9$ equal to 0? Yes! Therefore the solution is correct!

c. Using the Square Root Property

Solve $x^2 - 9 = 0$.

Isolate the squared term

$x^2 = 9$

and use the *Square Root Property*

$x = \pm\sqrt{9}$

then $x = 3$ or $x = -3$.

Problem 10:

If $3x^2 - 2x + 7 = 0$, then $\left(x - \frac{1}{3}\right)^2$ is equal to what number?

This is tricky! The only option we have is to see if the quantity $\left(x - \frac{1}{3}\right)^2$ is hiding somewhere in $3x^2 - 2x + 7 = 0$.

First of all, let's multiply out $\left(x - \frac{1}{3}\right)^2$ to get

$$\left(x - \tfrac{1}{3}\right)^2 = \left(x - \tfrac{1}{3}\right)\left(x - \tfrac{1}{3}\right)$$
$$= x^2 - \tfrac{2}{3}x + \tfrac{1}{9}$$

Certainly, we could write $3x^2 - 2x + 7 = 0$ as $3(x^2 - \frac{2}{3}x \quad) + 7 = 0$. That is, we factored a 3 out of the first two terms.

The terms enclosed in the parentheses now look like the first two terms of $x^2 - \frac{2}{3}x + \frac{1}{9}$.

So how could be insert a $\frac{1}{9}$ without changing the value of the quadratic equation?

Watch this! If we were to insert a $\frac{1}{9}$ into the parentheses then actually we changed the value of the left side by $3(\frac{1}{9})$ because there is a 3 in front of the parentheses.

However, if we were to do the following we would not have changed the value of the left side at all:

$$3(x^2 - \frac{2}{3}x + \frac{1}{9}) + 7 - 3(\frac{1}{9}) = 0$$

Note that we both added and subtracted $3(\frac{1}{9})$ on the left side. In effect, nothing happened except a change in appearance!

We can further write

$$3(x^2 - \frac{2}{3}x + \frac{1}{9}) + 7 - \frac{1}{3} = 0$$

$$3(x - \frac{1}{3})^2 + \frac{20}{3} = 0$$

Again, we have not changed the value of the left side of the original equation $3x^2 - 2x + 7 = 0$, just its appearance!

Now, given $3(x - \frac{1}{3})^2 + \frac{20}{9} = 0$, we can solve for the quantity $\left(x - \frac{1}{3}\right)^2$ as follows:

$$3(x - \frac{1}{3})^2 = -\frac{20}{3}$$

$$\frac{1}{3}[3(x - \frac{1}{3})^2] = -\frac{20}{3}(\frac{1}{3})$$

and we find $\left(x - \frac{1}{3}\right)^2 = -\frac{20}{9}$

Problem 11:

The monthly profit, ***P***, in thousands of dollars, of a company can be estimated by the formula $P = -3x^2 + 30x + 12$, where ***x*** is the number of units produced and sold per month. Find the profit when 5 units are sold in one month.

$$P = -3(5)^2 + 30(5) + 12$$

$$P = 87$$

That is, when 5 units are sold in one month the profit is *$87,000* because the formula is given in thousands of dollars.

Problem 12:

A projectile is shot upward. It's distance s above the ground after t seconds is $s = -16t^2 + 400t$. Obviously, what goes up must come down! Calculate the time it takes for the projectile to return to the ground.

In physics, it is assumed that when $s = 0$ the object is on the ground.

Thus, $-16t^2 + 400t = 0$ will give us the time at which the projectile hits the ground.

Let's solve for t by factoring and then using the *Zero Product Principle* as follows:

$$-16t(t-25) = 0$$

Then

$$-16t = 0$$
$$t = 0$$

This tells us that at $t = 0$, the object is on the ground!

$$t - 25 = 0$$
$$t = 25$$

From this calculation we find that the projectile is on the ground again after *25 seconds*.