DETAILED SOLUTIONS AND CONCEPTS
INTRODUCTION TO IRRATIONAL AND IMAGINARY NUMBERS

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PLEASE NOTE THAT YOU CANNOT USE A CALCULATOR ON THE ACCUPLACER - ELEMENTARY ALGEBRA TEST! YOU MUST BE ABLE TO DO THE FOLLOWING PROBLEMS WITHOUT A CALCULATOR!

Natural Numbers

The numbers used for counting. That is, the numbers ***{1, 2, 3, 4, ...}***.

Integers

The numbers ***{... , - 4, -3, -2, -1, 0, 1, 2, 3, 4, ...}***.

Whole Numbers or Nonnegative Integers

The numbers ***{0, 1, 2, 3, 4, ...}***.

Rational Numbers

Any type of number that can be written as the quotient of two *Integers*. This includes all terminating and repeating decimals, fractions, and the *Integers*.

Irrational Numbers

Any type of number that **cannot** be written as the quotient of two *Integers*. They are non-terminating decimal numbers.

Most irrational numbers result from findings roots of numbers that are NOT perfect powers. However, there are also some irrational numbers that occur naturally, such as the number π (approximately 3.14) and the number ***e*** (approximately 2.72).

NOTE: The results when natural numbers are squared, cubed, or raised to any power are also referred to as **perfect powers**! The root of a perfect power is a natural number.

Real Numbers

The *Real Numbers* include all of the *Rational* and *Irrational Numbers*.

Imaginary Numbers

Most imaginary numbers result from findings roots of negative numbers given an **EVEN index only.** A purely imaginary number is represented by the letter i and i is equal to $\sqrt{-1}$. Please note that given an odd index, roots of negative numbers result in rational or irrational numbers.

> NOTE: There is no *real number* that can be squared to get a result of -1. Therefore, the solution to $\sqrt{-1}$ only exists in our imagination.

Finding Rational, Irrational, and Imaginary Numbers

Problem 1:

If possible, find the square root of 144.

$\sqrt{144} = 12$, where 12 is a terminating decimal, specifically an integer, which is a rational number.

> Remember that $12(12)$ does equal 144 !!!

Problem 2:

If possible, find the **cube root** of -27.

$\sqrt[3]{-27} = -3$, where -3 is a terminating decimal, specifically an integer, which is a rational number.

Remember that $-3(-3)(-3)$ does equal -27 !!!

Problem 3:

If possible, find the **cube root** of 144 rounded to three decimal places.

Here we notice that the number 144 is not a perfect cube! That is, we CANNOT find a number written as the quotient of two integers that, when cubed, results in 144!

> **NOTE: For a problem like this, the ACCUPLACER test will make a calculator available to you!**

According to the calculator $\sqrt[3]{144} \approx 5.241482788$, where 5.241482788 is a non-terminating decimal, which is an irrational number.

Please note that the calculator eventually rounds to a certain number of decimal places. That does not mean that the decimal terminated.

Since we are asked to round the answer to three decimal places, we find $\sqrt[3]{144}$ to be approximately equal to 5.241.

Problem 4:

If possible, find the **cube root** of ***-7*** rounded to three decimal places.

Again, ***-7*** is not a perfect cube.

According to the calculator $\sqrt[3]{-7} \approx -1.912931183$, where ***-1.912931183*** is a non-terminating decimal, which is an irrational number. Note that the index is odd, therefore, the root is NOT imaginary!

> We CANNOT find a number written as the quotient of two integers that, when cubed, results in -7.

Since we are asked to round the answer to three decimal places, we find $\sqrt[3]{-7}$ to be approximately equal to ***-1.913***.

Problem 5:

Given the number ***81***, find its **square root**, **cube root**, and **4th root**, if possible. Round to three decimal places, if necessary.

square root: $\sqrt{81} = 9$ ... a rational number because ***9(9) = 81***

cube root: $\sqrt[3]{81} \approx 4.326748711$ ... an irrational number because we CANNOT find a number written as the quotient of two integers that, when cubed, results in 81.

Since we are asked to round the answer to three decimal places, we find $\sqrt[3]{81}$ to be approximately equal to ***4.327***.

4th root: $\sqrt[4]{81} = 3$ ... a rational number because ***3(3)(3)(3) = 81***

Problem 6:

If possible, find the **square root** of ***-81***.

$\sqrt{-81}$ is an imaginary number because the INDEX IS EVEN and the radicand is negative.

There is no *real number* that can be squared to get a result of ***-81***. Therefore, the solution to $\sqrt{-81}$ only exists in our imagination.

Problem 7:

If possible, find the square root of ***-3***.

$\sqrt{-3}$ is an imaginary number because the INDEX IS EVEN and the radicand is negative.

There is no *real number* that can be squared to get a result of ***-3***. Therefore, the solution to $\sqrt{-3}$ only exists in our imagination.

Problem 8:

Given the number ***-64***, find its **square root** and **cube root**, if possible.

square root: $\sqrt{-64}$ ... an imaginary number because the index is even

cube root: $\sqrt[3]{-64} = -4$ ... a rational number because the index is odd and ***-4(-4)(-4) = -64***

Simplifying Radical Expressions

Please note that the word "simplify" takes on many meanings in mathematics. Often you must figure out its meaning from the mathematical expression you are asked to "simplify." Here are are asked to "simplify" instead of to finding the root of a number.

Before we begin, we must know that radical expressions can also be written as exponential expression. Following are the conversions:

$$\sqrt[root]{x^{power}} = x^{power/root}$$

Furthermore, $\sqrt[root]{x^{power}}$ is equivalent to $\left(\sqrt[root]{x}\right)^{power}$

Problem 9:

Write $\sqrt[4]{81}$ as an exponential expression and simplify.

$81^{\frac{1}{4}}$ and $\sqrt[4]{81} = 3$. As you can see the index ***4*** becomes the denominator of a fractional power with a numerator of ***1***.

Problem 10:

Write $\sqrt[3]{27}$ as an the exponential expression and simplify.

$27^{\frac{1}{3}}$ and $\sqrt[3]{27} = 3$. As you can see the index ***3*** becomes the denominator of a fractional power with a numerator of ***1***.

Problem 11:

Write $\sqrt{9}$ as an exponential expression and simplify.

$9^{\frac{1}{2}}$ and $\sqrt{9} = 3$. As you can see the index ***2*** (it is customary to not write it) becomes the denominator of a fractional power with a numerator of ***1***.

Problem 12:

Write $\sqrt{y^{10}}$ as an exponential expression and simplify.

$y^{\frac{10}{2}} = y^5$ As you can see the index ***2*** (it is customary to not write it) becomes the denominator of a fractional power with a numerator of ***10***. and then we can reduce the exponential fraction.

Problem 13:

Write $\sqrt{\frac{x^2}{y^6}}$ as an exponential expression and simplify.

$\left(\frac{x^2}{y^6}\right)^{\frac{1}{2}}$ As you can see the index ***2*** becomes the denominator of a fractional power with a numerator of ***1***.

Using one of the *Laws of Exponents* we can further simplify to get the following:

$$\frac{(x^2)^{\frac{1}{2}}}{(y^6)^{\frac{1}{2}}} = \frac{x}{y^3}$$

Problem 14:

Write $\sqrt[4]{16b^8}$ as an exponential expression and simplify.

$(16b^8)^{\frac{1}{4}}$ As you can see the index ***4*** becomes the denominator of a fractional power with a numerator of ***1***.

Using one of the *Laws of Exponents* we can further simplify to get the following:

$16^{\frac{1}{4}}b^{\frac{8}{4}}$ which can be further simplified to $2b^2$.

NOTE: It is expected that you have permanently committed to memory the following values:

$2^2 = 4$ $2^3 = 8$ $2^4 = 16$ $2^5 = 32$ $2^6 = 64$

$3^2 = 9$ $3^3 = 27$ $3^4 = 81$

$4^2 = 16$ $4^3 = 64$

$5^2 = 25$ $5^3 = 125$

$6^2 = 36$ $7^2 = 49$ $8^2 = 64$ $9^2 = 81$ $10^2 = 100$

$11^2 = 121$ $12^2 = 144$ $13^2 = 169$

$14^2 = 196$ $15^2 = 225$ $16^2 = 256$

$17^2 = 289$ $18^2 = 324$ $19^2 = 361$ $20^2 = 400$

Problem 15:

Write $\sqrt[3]{27x^2y^6}$ as an exponential expression and simplify

$(27x^2y^6)^{\frac{1}{3}}$ As you can see the index 3 becomes the denominator of a fractional power with a numerator of 1.

Using one of the *Laws of Exponents* we can further simplify to get the following:

$27^{\frac{1}{3}}x^{\frac{2}{3}}y^{\frac{6}{3}}$ which can be further simplified to $3x^{\frac{2}{3}}y^2$.

Problem 16:

Write $\sqrt[3]{x^2}$ as an exponential expression.

$x^{\frac{2}{3}}$ As you can see the index 3 becomes the denominator of a fractional power with a numerator of 2.

Problem 17:

Write $\sqrt[4]{a^3}$ as an exponential expression.

$a^{\frac{3}{4}}$ As you can see the index 4 becomes the denominator of a fractional power with a numerator of 3.

Problem 18:

Write $\sqrt{a^3}$ as an exponential expression.

$a^{\frac{3}{2}}$ As you can see the index ***2*** becomes the denominator of a fractional power with a numerator of ***3***.