DETAILED SOLUTIONS AND CONCEPTS - OPERATIONS ON IMAGINARY NUMBERS

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PLEASE NOTE THAT YOU CANNOT USE A CALCULATOR ON THE ACCUPLACER - ELEMENTARY ALGEBRA TEST! YOU MUST BE ABLE TO DO THE FOLLOWING PROBLEMS WITHOUT A CALCULATOR!

Imaginary Numbers

- Most imaginary numbers result from findings roots of negative numbers given an **EVEN index only.** A purely imaginary number is represented by the letter ***i*** and ***i*** is equal to $\sqrt{-1}$. Please note that given an odd index, roots of negative numbers result in rational or irrational numbers.

  NOTE: There is no *real number* that can be squared to get a result of ***-1***. Therefore, the solution to $\sqrt{-1}$ only exists in our imagination.

- When we encounter the square root of a negative number, it is customary to take the negative sign out of the radical and convert it to the letter ***i*** as follows:

  $\sqrt{-a} = i\sqrt{a}$

- Furthermore, $i^2 = -1$

Complex Numbers

Complex Numbers are of the form ***a + bi***, where ***a*** is a real number and ***bi*** a purely imaginary number with coefficient ***b***. All real numbers can be written in complex form.

For example, ***3 + 0i***, ***-2.34 + 0i***, etc.

On the other hand, ***3 + 2i*** or ***-2.34 - 5.1i*** are complex number containing an imaginary part and are therefore called imaginary numbers.

Problem 1:

Simplify $\sqrt{-81}$, if possible, and write in terms of ***i***.

$\sqrt{-81}$ is an imaginary number because the INDEX IS EVEN and the radicand is negative.

There is no *real number* that can be squared to get a result of ***-81***. Therefore, the solution to $\sqrt{-81}$ only exists in our imagination.

When we encounter the square root of a negative number, it is customary to take the negative sign out of the radicand and convert it to the letter "i" as follows:

$\sqrt{-81} = i\sqrt{81}$. **There is an assumed multiplication sign between the number *i* and the radical expression.**

Since the number ***81*** is a perfect square, we can further write
$\sqrt{-81} = i\sqrt{81} = 9i$.

NOTE: It is customary to write the factor ***i*** AFTER a number once the radical sign is eliminated.

Problem 2:

Write $\sqrt{-3}$ in terms of ***i***.

$\sqrt{-3}$ is an imaginary number because the INDEX IS EVEN and the radicand is negative.

There is no *real number* that can be squared to get a result of ***-3***. Therefore, the solution to $\sqrt{-3}$ only exists in our imagination.

However, we can simplify $\sqrt{-3}$ by writing $\sqrt{-3} = i\sqrt{3}$.

NOTE: It is customary to write the ***i*** in front of the radical!

Sometimes, we want to change the radical expression to a decimal approximation (remember it is a non-terminating decimal) in which case we write

$$i\sqrt{3} \approx 1.73i$$

NOTE: It is customary to write the ***i*** AFTER a number once the radical sign is eliminated.

Problem 3:

Simplify $\sqrt{-64}$, if possible, and write in terms of ***i***.

$\sqrt{-64}$ is an imaginary number because the INDEX IS EVEN and the radicand is negative.

There is no *real number* that can be squared to get a result of ***-64***. Therefore, the solution to $\sqrt{-64}$ only exists in our imagination.

However, we can simplify by writing $\sqrt{-64} = i\sqrt{64} = 8i$.

> NOTE: It is customary to write the factor ***i*** AFTER a number once the radical sign is eliminated.

Adding and Subtracting Complex Numbers

- Add or subtract the real parts.
- Add or subtract the coefficients of the imaginary parts.

Problem 4:

Add ***(3 + 6i) + (9 - 2i)***.

> NOTE: When you carry out an arithmetic operation on complex numbers, you must enclose them in parentheses!

We can rewrite this as follows:

$$3 + 9 + 6i - 2i = 12 + (6 - 2)i$$

$$= 12 + 4i$$

Problem 5:

Subtract ***(2 + 7i) - (8 - i)***.

In this case, we MUST observe the minus sign in front of the parentheses.

We first must write ***2 + 7i - 8 + i***.

The we combine "like" terms to get ***-6 + 8i*** .

> Please note that ***i*** has a coefficient of ***1*** which is usually not written, but must be used in addition and subtraction.

Multiplying Complex Numbers

Multiplying complex numbers uses procedures similar to multiplying polynomials!

Problem 6:

Multiply ***7(3i)***.

Here we multiply the coefficients to get ***21i***.

Problem 7:

Multiply ***7i(3i)***.

Here we multiply the coefficients and the imaginary numbers to get $21i^2$.

Since we know that $i^2 = -1$, we can state

$21i^2 = 21(-1) = -21$

Problem 8:

Multiply ***(2 + 7i)(8 - 3i)***.

Use the ***FOIL*** process to multiply ***(2 + 7i)(8 - 3i)***.

F O I L

then $16 - 6i + 56i - 21i^2$

Since we know that $i^2 = -1$, we can write

$16 - 6i + 56i - 21(-1) = 16 - 6i + 56i + 21$

and finally we can combine like terms to get

$37 + 50i$

Problem 9:

Factor the *Sum of Squares* $x^2 + 4$.

Now we know that the *Difference of Squares* $x^2 - 4$ is factored into $(x - 2)(x + 2)$.

The *Sum of Squares*, on the other hand is factored into $(x - 2i)(x + 2i)$.

Check:

Use FOIL to multiply $(x - 2i)(x + 2i)$.

F O I L

then $x^2 + 2i - 2i - 4i^2$

Since we know that $i^2 = -1$, we can write

$x^2 + 2i - 2i - 4(-1)$

and multiplying and combining like terms will result in $x^2 + 4$.

Rationalizing a Denominator containing a Complex Number

- Multiply the denominator by its conjugate ***.
- To preserve the value of the fraction, multiply the numerator by the same number.
- Simplify all and write the number in the form ***a + bi***.

*** The conjugate of a complex number ***a + bi*** is the complex number ***a - bi***.

NOTE: In Steps 1 and 2 above, we have actually multiplied the fraction by an equivalent of the number 1!

Problem 10:

Rationalize the denominator of $\frac{4+i}{3-i}$ and write in standard form ***a + bi***.

First, we will multiply both the numerator and the denominator by ***3 + i***, which is the conjugate of the denominator.

$$\frac{(4+i)(3+i)}{(3-i)(3+i)}$$

Next, we will use the FOIL method to multiply the complex numbers in the numerator. Observe that the denominator contains a *Difference of Squares*!

$$\frac{12+4i+3i+i^2}{9-i^2}$$

Since we know that $i^2=-1$, we can write

$$\frac{12+7i-1}{9-(-1)}=\frac{11+7i}{10}$$

and finally, we find that we can express $\frac{4+i}{3-i}$ in standard form as $\frac{11}{10}+\frac{7}{10}i$.

Problem 11:

Rationalize the denominator of $\frac{6-i}{4+i}$ and write in standard form ***a + bi***.

First, we will multiply both the numerator and the denominator by ***4 - i***, which is the conjugate of the denominator.

$$\frac{(6-i)(4-i)}{(4+i)(4-i)}$$

Next, we will use the FOIL method to multiply the complex numbers in the numerator. Observe that the denominator contains a *Difference of Squares*!

$$\frac{24 - 6i - 4i + i^2}{16 - i^2}$$

Since we know that $i^2 = -1$, we can write

$$\frac{24 - 10i - 1}{16 - (-1)} = \frac{23 - 10i}{17}$$

and finally, we find that we can express $\frac{6 - i}{4 + i}$ in standard form as $\frac{23}{17} - \frac{10}{17}i$.

Problem 12:

Rationalize the denominator of $\frac{-6 - 2i}{-4 + 2i}$ and write in standard form ***a + bi***.

First, we will multiply both the numerator and the denominator by ***-4 - 2i***, which is the conjugate of the denominator.

$$\frac{(-6 - 2i)(-4 - 2i)}{(-4 + 2i)(-4 - 2i)}$$

Next, we will use the FOIL method to multiply the complex numbers in the numerator. Observe that the denominator contains a *Difference of Squares*!

$$\frac{24 + 12i + 8i + 4i^2}{16 - 4i^2}$$

Since we know that $i^2 = -1$, we can write

$$\frac{24 + 20i + 4(-1)}{16 - 4(-1)} = \frac{20 + 20i}{20}$$

and $\frac{20}{20} + \frac{20i}{20} = 1 + i$.

Finally, we find that we can express $\frac{-6 - 2i}{-4 + 2i}$ in standard form as ***1 + i***.