DETAILED SOLUTIONS AND CONCEPTS - FACTORING POLYNOMIAL EXPRESSIONS

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PLEASE NOTE THAT YOU CANNOT USE A CALCULATOR ON THE ACCUPLACER - ELEMENTARY ALGEBRA TEST! YOU MUST BE ABLE TO DO THE FOLLOWING PROBLEMS WITHOUT A CALCULATOR!

Factoring - It actually means to "divide out.". When you are asked to "factor" a polynomial expression, you are actually told to write the expression as a product of factors. Often you are asked to factor "relative to the integers." This means that your factors cannot include rational, irrational, or imaginary numbers.

A. Factoring out the Greatest Common Factor.

- Write each number in an expression in prime factored form.
- List each prime number that is a factor of every number in the list. Multiply these numbers. This is our greatest common numeric factor.

  Note: Most of the time you can find the common numeric factor by "casual observation."

- Find the variable with the smallest exponent. This is our greatest common variable factor.

Problem 1:

Given the polynomial expression $3x^7 - 18x^3 + 9x^2$, factor out the **Greatest Common Factor.**

We can easily find the greatest common numeric factor by "casual observation." It is 3. The greatest common variable factor is x^2 .

Therefore, $3x^7 - 18x^3 + 9x^2$ can be written as $3x^2(x^5 - 6x + 3)$, which is a product of two factors. One is a monomial and the other one a trinomial.

Problem 2:

Given the polynomial expression $15x^2 - 5x$, factor out the **Greatest Common Factor.**

We can easily find the greatest common numeric factor by "casual observation." It is 5. The greatest common variable factor is x .

Therefore, $15x^2 - 5x$ can be written as $5x (3x - 1)$, which is a product of two factors.

Problem 3:

Given the polynomial expression $8x^2 + 4$, factor out the **Greatest Common Factor.**

We can easily find the greatest common numeric factor by "casual observation." It is 4. There is no greatest common variable factor.

Therefore, $8x^2 + 4$ can be written as $4(2x^2 + 1)$, which is a product of two factors.

Problem 4:

Given the polynomial expression $6x^3 - 8x^2$, factor out the **Greatest Common Factor.**

We can easily find the greatest common numeric factor by "casual observation." It is 2. The greatest common variable factor is x^2 .

Therefore, $6x^3 - 8x^2$ can be written as $2x^2 (3x - 4)$, which is a product of two factors.

Problem 5:

Given the polynomial expression $kx^2 - ktx + 3x - 3t$, where k and t are constants, factor out the **Greatest Common Factor** in the two middle terms.

The two middle terms $-ktx$ and $3x$ do not have a greatest common numeric factor. The greatest common variable factor is x .

Therefore, $kx^2 - ktx + 3x - 3t$ can be written as follows:

$kx^2 - (kt - 3)x - 3t$ or

$kx^2 + (-kt + 3)x - 3t$ or

$kx^2 + (3 - kt)x - 3t$

Problem 6:

Given the polynomial expression $2x(x - 3) - (x - 3)$, factor out the **Greatest Common Factor**.

This expression has two terms and both contain the factor $x - 3$. This is the Greatest Common Factor and we will factor it out as follows:

$$2x(x - 3) - (x - 3) = (x - 3)(2x - 1)$$

Please note that "factoring" means to "divide out". When we divide $x - 3$ out of the first term, we are left with a factor of $2x$; and when we divide $x - 3$ out of the second term we are left with a factor of -1.

Problem 7:

Given the polynomial expression $25q^2(m + 1)^2 - 15q(m + 1)^2 + 5(m + 1)^2$, factor out the **Greatest Common Factor**.

This expression has three terms and all contain the factor $5(m + 1)^2$. This is the Greatest Common Factor and we will factor it out as follows.:

$$5(m + 1)^2 (5q^2 - 3q + 1)$$

Please note that "factoring" means to "divide out". When we divide $5(m + 1)^2$ out of the first term, we are left with a factor of $5q^2$; when we divide $5(m + 1)^2$ out of the second term we are left with a factor of $-3q$; and when we divide out of the third term we are left with a factor of 1.

Problem 8:

Given the polynomial expression $15x^2(r + 3)^3 - 33x^2(r + 3)^2$, factor out the **Greatest Common Factor**.

This expression has three terms and all contain the factor $3x^2(r + 3)^2$. This is the Greatest Common Factor and we will factor it out as follows.:

$$3x^2(r + 3)^2 [5(r + 3) - 11] = 3x^2(r + 3)^2(5r + 15 - 11)$$

$$= 3x^2(r + 3)^2(5r + 4)$$

Please note that "factoring" means to "divide out". When we divide $3x^2(r + 3)^2$ out of the first term, we are left with a factor of $5(r + 3)$; and when we divide $3x^2(r + 3)^2$ out of the second term we are left with a factor of 11. In this case, we were even able to combine like terms in the final product! This type of factoring is often necessary in a calculus course!

B. Factoring relative to the integers using the Grouping Method.

- Collect the terms in an expression into two groups.
- Factor out the greatest common factor from each group. You should end up with an identical multinomial factor in each term. If not, regroup, then factor out the greatest common factor again. If you still don't end up with an identical multinomial factor in each term, the expression is NOT factorable by grouping.
- Factor out the identical multinomial factor from each term.

Problem 9:

Try to factor the polynomial expression $x^3 - 4x^2 + 2x - 8$ relative to the integers using the **Grouping Method**.

Let's form two groups as follows: $(x^3 - 4x^2) + (2x - 8)$

Please note that these were the most convenient groups. It just so happens that when we factor common factors out of both, we have an identical binomial factor in each term!

$x^2(x - 4) + 2(x - 4)$ **Note that we now have an identical binomial factor in each term!**

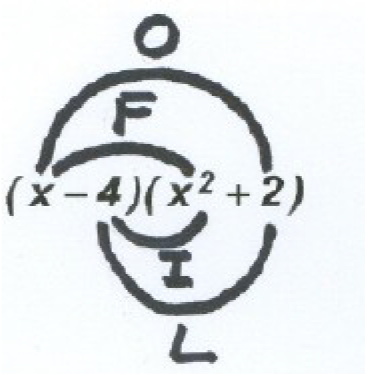
Lastly, we will factor out the identical binomial factor $(x - 4)$ as follows:

$(x - 4)(x^2 + 2)$

Therefore, $x^3 - 4x^2 + 2x - 8$ can be written as $(x - 4)(x^2 + 2)$, which is a product of two factors. Both are binomials.

Let's use **FOIL** to check the factoring process!

> NOTE: **FOIL** stands for First, Outer, Inner, Last and refers to how the terms in two binomials are multiplied out. **FOIL** is a shortcut for the *Distributive Property*!



First: $x \cdot x^2 = x^3$ Outer: $x \cdot 2 = 2x$

Inner: $-4 \cdot x^2 = -4x^2$ Last: $-4 \cdot 2 = -8$

Using **FOIL**, we can indeed see that $(x- 4)(x^2 +2) = x^3 - 4x^2 + 2x - 8$.

Problem 10:

Try to factor the polynomial expression $3x^3 + 2x^2 - 6x + 2$ relative to the integers using the **Grouping Method**.

Let's form two groups as follows:

$(3x^3 + 2x^2) - (6x - 2)$ **Please note that the sign between 6x and 2 changed. This is due to factoring out a minus!**

Now we'll factor out the greatest common factor from each group.

$x^2(3x + 2) - 2(3x - 1)$

Note that we did not get two identical factors in each term. Before we give up, let's try a different set of groups.

By rewriting $3x^3 + 2x^2 - 6x + 2$ to $3x^3 - 6x + 2x^2 + 2$ we can group as follows:

$(3x^3 - 6x) + (2x^2 + 2)$

Again we factor out the greatest common factor from each group.

$3x(x^2 - 2) + 2(x^2 + 1)$

Since we did not get two identical factors in each term again, we concede that we cannot use the *Grouping Method* to factor this polynomial relative to the integers. Please be aware that not all polynomial expressions are factorable, although, in this case we have not tried more advanced method beyond the scope of this lecture!

C. Factoring the trinomial $ax^2 + bx + c$ relative to the integers. Assume $a = 1$.

- Using "educated guessing", find two integers whose product equals c and whose sum equals b. These integers are always factors of c. If this is not possible, the trinomial cannot be factored relative to the integers.
- Form a product using two binomials containing the variable x and each of the integers found.

Problem 11:

Try to factor the trinomial $x^2 - 5x + 6$ relative to the integers.

Given the general trinomial $ax^2 + bx + c$, we find that $b = -5$ and $c = 6$.

YES, actually using some educated guessing, we need to find two integers whose **product** equals 6 and whose **sum** equals -5. However, the guessing is simplified because the integers MUST be factors of c. In this case of 6.

Let's write down the factors of 6:

$$-6, -3, -2, -1, 1, 2, 3, 6$$

It seems that the two integers we need are -2 and -3 since $-2(-3) = 6$ and $-2 + (-3) = -5$.

Therefore, $x^2 - 5x + 6$ factored relative to the integers becomes the product $(x - 2)(x - 3)$. As a beginner you might want to use **FOIL** to convince yourself that you have indeed found the correct factors.

Problem 12:

Try to factor the trinomial $x^2 + 5x + 6$ relative to the integers.

Given the general trinomial $ax^2 + bx + c$, we find that $b = 5$ and $c = 6$.

Again, using some educated guessing, we need to find two integers whose **product** equals 6 and whose **sum** equals 5.

Let's write down the factors of 6:

$$-6, -3, -2, -1, 1, 2, 3, 6$$

The two integers we need this time are 2 and 3 since $2(3) = 6$ and $2 + 3 = 5$.

Therefore, $x^2 + 5x + 6$ factored relative to the integers becomes the product $(x + 2)(x + 3)$. As a beginner you might want to use **FOIL** to convince yourself that you have indeed found the correct factors.

Problem 13:

Try to factor the trinomial $x^2 - 5x - 6$ relative to the integers.

Given the general trinomial $ax^2 + bx + c$, we find that $b = -5$ and $c = -6$.

Let's write down the factors of -6:

$-6, -3, -2, -1, 1, 2, 3, 6$

Remember, we need to find two integers whose **product** equals -6 and whose **sum** equals -5.

The two integers we need are -6 and 1 since $-6(1) = -6$ and $-6 + 3 = -5$.

Therefore, $x^2 - 5x - 6$ factored relative to the integers becomes the product $(x - 6)(x + 1)$. As a beginner you might want to use **FOIL** to convince yourself that you have indeed found the correct factors.

Problem 14:

Try to factor the trinomial $x^2 + 5x - 6$ relative to the integers.

Given the general trinomial $ax^2 + bx + c$, we find that $b = 5$ and $c = -6$.

Let's write down the factors of -6:

$-6, -3, -2, -1, 1, 2, 3, 6$

We need to find two integers whose **product** equals -6 and whose **sum** equals 5.

The two integers we need are 6 and -1 since $6(-1) = -6$ and $6 + (-1) = 5.$

Therefore, $x^2 + 5x - 6$ factored relative to the integers becomes the product $(x + 6)(x - 1)$. As a beginner you might want to use **FOIL** to convince yourself that you have indeed found the correct factors.

Problem 15:

Try to factor the trinomial $x^2 - 10x + 25$ relative to the integers.

Given the general trinomial $ax^2 + bx + c$, we find that $b = -10$ and $c = 25$.

Let's write down the factors of 25:

-25, -5, -1, 1, 5, 25

We need to find two integers whose **product** equals ***25*** and whose **sum** equals ***-10***.

The two integers we need are ***-5*** and ***-5*** since ***-5(-5) = 25*** and ***-5 + (-5) = -10.***

Therefore, $x^2 - 10x + 25$ factored relative to the integers becomes the product
$(x - 5)(x - 5) = (x - 5)^2$ **This is called a *Perfect Square Trinomial*.**

Problem 16:

Try to factor the trinomial $x^2 + 6x + 9$ relative to the integers.

Given the general trinomial $ax^2 + bx + c$, we find that ***b = 6*** and ***c = 9***.

Let's write down the factors of ***9***:

-9, -3, -1, 1, 3, 9

We need to find two integers whose **product** equals ***9*** and whose **sum** equals ***6***.

The two integers we need are ***3*** and ***3*** since ***3(3) = 9*** and ***3 + 3 = 6***.

Therefore, $x^2 + 6x + 9$ factored relative to the integers becomes the product
$(x + 3)(x + 3) = (x + 3)^2$ **This is also called a *Perfect Square Trinomial*.**

Problem 17:

Try to factor the trinomial $x^2 + 2x + 4$ relative to the integers.

Given the general trinomial $ax^2 + bx + c$, we find that ***b = 2*** and ***c = 4***.

Let's write down the factors of ***4***:

-4, -2, -1, 1, 2, 4

We need to find two integers whose **product** equals ***4*** and whose **sum** equals ***2***.

This trinomial cannot be factored relative to the integers:

Using ***2*** and ***2*** we get ***2(2) = 4***, but $2 + 2 \neq 2$.

Using ***-2*** and ***-2*** we get ***-2(-2) = 4***, but $-2 + (-2) \neq 2$.

Using ***4*** and ***1*** we get ***4(1) = 4***, but $4 + 1 \neq 2$.

Using ***-4*** and ***-1*** we get ***-4(-1) = 4***, but $-4 + (-1) \neq 2$.

D. Factoring the Trinomial $ax^2 + bx + c$ relative to the Integers. Assume $a \neq 1$.

- Find two integers whose product equals ***ac*** and whose sum equals ***b***. These integers are always factors of ***ac***. If this is not possible, the trinomial cannot be factored relative to the integers.
- Replace the middle term ***bx*** in $ax^2 + bx + c$ with the sum of the integers.
- Use the factoring by grouping method to find a product of two binomials.

Problem 18:

Try to factor the trinomial $2x^2 + 7x + 6$ relative to the integers.

Given the general trinomial $ax^2 + bx + c$, we find that $a = 2$, $b = 7$ and $c = 6$.

We need to find two integers whose **product** equals $2(6) = 12$ and whose **sum** equals 7. However, the guessing is simplified because the integers MUST be factors of ***ac***. In this case of 12.

Let's write down the factors of 12:

$$-12, -6, -4, -3, -2, -1, 1, 2, 3, 4, 6, 12$$

It seems that the two integers we need are 3 and 4 since $3(4) = 12$ and $3 + 4 = 7$.

Now we write $2x^2 + 7x + 6$ as follows:

$$2x^2 + 3x + 4x + 6$$

Next, we'll use the factoring by grouping method to get $(2x^2 + 3x) + (4x + 6)$.

Then, we factor the common factor out of both groups as follows:

$$x(2x + 3) + 2(2x + 3)$$

Lastly, we will factor out the identical binomial factor to get $(2x + 3)(x + 2)$.

Therefore, $2x^2 + 7x + 6$ factored relative to the integers becomes the product $(2x + 3)(x + 2)$. As a beginner you might want to use **FOIL** to convince yourself that you have indeed found the correct factors.

Problem 19:

Try to factor the trinomial $6x^2 + x - 2$ relative to the integers.

Given the general trinomial $ax^2 + bx + c$, we find that $a = 6$, $b = 1$ and $c = -2$.

We need to find two integers whose **product** equals $6(-2) = -12$ and whose **sum** equals 1.

However, the guessing is simplified because the integers MUST be factors of ***ac***. In this case of ***12***.

Let's write down the factors of ***-12***:

-12, -6, -4, -3, -2, -1, 1, 2, 3, 4, 6, 12

It seems that the two integers we need are ***-3*** and ***4*** since ***-3(4) = -12*** and ***-3 + 4 = 1***.

Now we write $6x^2 + x - 2$ as follows:

$$6x^2 - 3x + 4x - 2$$

Next, we'll use the factoring by grouping method to get $(6x^2 - 3x) + (4x - 2)$.

Then, we factor the common factor out of both groups as follows:

$$3x(2x - 1) + 2(2x - 1)$$

Lastly, we will factor out the identical binomial factor to get $(2x - 1)(3x + 2)$.

Therefore, $6x^2 + x - 2$ factored relative to the integers becomes the product $(2x - 1)(3x + 2)$. As a beginner you might want to use **FOIL** to convince yourself that you have indeed found the correct factors.

E. Factoring "special" polynomials relative to the Integers.

- *Difference of Squares:* $x^2 - a^2 = (x - a)(x + a)$
- *Difference of Cubes:* $x^3 - a^3 = (x - a)(x^2 + ax + a^2)$
- *Sum of Cubes*: $x^3 + a^3 = (x + a)(x^2 - ax + a^2)$

Please note that a *Sums of Squares* cannot be factored relative to the integers. The factors of *Sums of Squares* are imaginary! This will be discussed at a later time!

Problem 20:

Factor the following "special" polynomials relative to the integers.

(a) $x^2 - 9$

Using the *Difference of Squares* formula with ***a = 3***, we can factor as follows:

$$(x - 3)(x + 3)$$

(b) $k^2 - m^2$

Using the *Difference of Squares* formula with $a = m$, we can factor as follows:

$(k - m)(k + m)$

(c) $x^3 - 8$

Using the *Difference of Cubes* formula with $a = 2$, we can factor as follows:

$(x - 2)(x^2 + 2x + 4)$

(d) $x^3 + 125$

Using the *Sum of Cubes* formula $a = 5$, we can factor as follows:

$(x + 5)(x^2 - 5x + 25)$

E. Factoring polynomials that are "quadratic" in form relative to the integers.

Problem 21:

Factor $x^4 - 8x^2 - 9$ relative to the integers.

Sometimes, you can encounter polynomial expressions that are "quadratic" in form. That is, one exponent is exactly twice as large as the other exponent!

We can rewrite the first variable as follows using the Laws of Exponents:

$$(x^2)^2 - 8x^2 - 9$$

Please note that we could now say the following:

Let $x^2 = a$ and we can rewrite the given polynomial expressions as $a^2 - 8a - 9$.

We know that we can factor $a^2 - 8a - 9$ into $(a - 9)(a + 1)$.

Therefore, we can factor the original expression as follows:

$$(x^2 - 9)(x^2 + 1)$$

We further recognize that the factor $x^2 - 9$ is a *Difference of Squares*, which can be factored into $(x - 3)(x + 3)$.

Now, the factor $x^2 + 1$ is a *Sum of Squares*. As noted above, *Sums of Squares* cannot be factored relative to the integers! Their factors are imaginary. This will be discussed at a later time

Finally, the original expression can be written as $(x-3)(x+3)(x^2+1)$!